**Real Estate Math Overview
and Introduction to the
HP 10bII+ Calculator**

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The purpose of this supplement is to provide an overview of the real estate mathematics required in BUSI 221, reviewing how to perform calculations for investments in real estate (and other assets). The BUSI 121 course or equivalent is a pre-requisite for BUSI 221, so this course has been prepared with the expectation that students already have mastered real estate mathematics foundations. However, we recognize that students may not have taken a math course in some time and need a refresher, or that their math instruction was organized differently or was based on a different financial calculator. Therefore, this supplement will review basic real estate math calculations using the Hewlett Packard (HP) 10bII+ calculator.

If you find this supplement difficult and have trouble with the math questions, then you should take this as a sign that you need to do more math review before you continue in BUSI 221. The calculations will build in complexity in this course, so if you are confused now, it will get worse later!

The first section of this supplement provides a brief overview of the HP 10bII+ calculator. Instructions in this course will show steps on this calculator. While you are welcome to use any calculator that is not both programmable and alphanumeric, if you elect to use a different calculator, you will need to figure out its operations on your own.

The second section of this supplement introduces a number of basic mortgage finance calculations including: conversion of an interest rate to an equivalent rate with a different compounding frequency, solving for constant payments, and calculating outstanding balances.

The third section of this supplement discusses alternative financing arrangements and investment analysis tools.

After studying this supplement, a student should be able to:

- differentiate between nominal and periodic rates of interest;
- understand how to use the HP 10bII+ calculator to solve basic mathematical problems;
- calculate future and present values for lump sums;
- calculate payments and outstanding balances for mortgage loans;
- calculate principal and interest portions of mortgages with varying payment frequencies and terms;
- calculate the market value of a fully or partially amortized vendor take-back mortgage;
- calculate the net present value (NPV) and internal rate of return (IRR) of a series of cash flows; and
- recognize the financial functions in Microsoft Excel.

Introduction to the Hewlett Packard (HP) 10bII+ Calculator

SHIFT Keys

SHIFT for statistical functions ⟶

SHIFT for financial functions ⟶
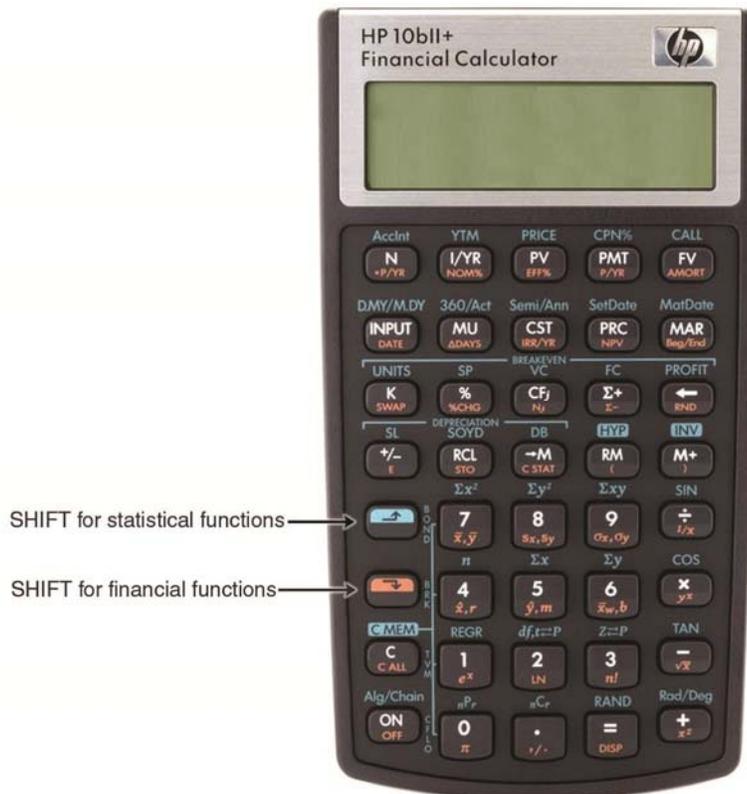
The HP 10bII+ has two ■ (shift) keys. One is orange (for financial functions), the other is blue (for statistical functions). To access the financial functions, students should always use the orange ■ key. All functions that are activated by the orange shift key are located in the lower half of each of the calculator keys, and are also labelled in orange.

BEG/END

Please be aware that the HP 10bII+ calculator has both Begin and End modes. The Begin mode is needed for annuity due calculations, or those which require payments to be made "in advance". For example, lease payments are generally made at the beginning of each month, not at the end. On the other hand, interest payments are almost always calculated at the end of each payment period, or "not in advance". These types of calculations each require a different setting on the calculator. When your calculator is set in Begin mode, the bottom of the display screen will show BEG. If BEG is not on your display screen, your calculator is in End mode, as there is no annunciator for this mode.

To switch between modes, press ■ BEG/END.

In this supplement, there are no calculations which require your calculator to be in Begin mode, so your calculator should be in End mode at all times. You should not see the BEG annunciator on your calculator's display. However, there are lease calculations in BUSI 331 that require this setting (later in the course).

Be careful of this, as the wrong setting will give you wrong answers!

Decimal Places On-Screen

The HP 10bII+ can display up 10 decimal places of accuracy. You may revise this by pressing the ORANGE ■, then DISP, and a number to show the number of decimal places you wish. For example, if you want to set the display to 9 decimal places, press the ORANGE ■ then DISP and then the 9 key. You will see 0.000000000 on your display screen as shown below:

Press	Display	Comments
■ DISP 9	0.000000000	

To display the maximum number of decimal places possible, you may wish to set your calculator to "floating decimal" as shown below:

Press	Display	Comments
■ DISP •	0	

The floating decimal notation will display very large or very small numbers in scientific notation or exponential function, (exponents) in order to show more decimal places of accuracy, e.g., 4/19 = 2.10526316E–1.

Press	Display	Comments
4 ÷ 19 =	2.10526316E–1	

This same value set to 9 decimal places changes the display to 0.210526316.

Press	Display	Comments
■ DISP 9	0.210526316	

The use of floating decimal places gives slightly more accuracy, but you have to be comfortable working with the exponents (E–n). This is your personal preference, as the final decimal place does not significantly affect

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Fixed Decimal Places

For ease of presentation, in each of the examples presented in this course, the calculator is programmed to display a "fixed decimal point" set to six decimal places. This is accomplished by turning the calculator ON, pressing the ORANGE ■ then DISP and then the 6 key. You will see 0.000000 on your display screen as shown below:

Press	Display	Comments
■ DISP 6	0.000000	

Note that on the display portion of future calculator steps, we will not show the display with zeros when they do not impact the result (and are mathematically insignificant).

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calculations.

Negative Numbers

The +/− key changes the sign of the displayed number to a negative or positive. To enter a negative number in your calculator, you must first enter the number and once it is showing on the display screen, you must press the +/− key to change it to a negative number.

Clearing Entries

To clear an unintentional numerical entry, you may use the ← key. This erases the last character entered.

The C key will clear the last number entered in a mathematical equation, so that you may correct it and continue with the equation. However, if you wish to clear the equation itself and start the calculation again, *you must press C twice!*

To clear all the stored values in the calculator, as well as the memory function:

Press	Display	Comments
■ C ALL	0	

Memory Keys

This calculator has a "constant memory". This means that whatever is stored in memory remains there until expressly changed (even when the calculator is turned off), unless the ■ C ALL function is used or the batteries are removed.

The calculator's memory keys are as follows:

→M   stores the number showing on the display screen in memory
RM   recalls a number from memory and displays it; the number remains stored in memory
M+   adds the number on the display screen to the number that is already stored in memory; the sum is retained in memory

Additional Function Keys

You may find numerous other function keys of use. For example, the % key converts a number expressed as a percent (e.g., 60%) to its decimal equivalent, i.e., 0.60.

Example 1 – Using the % Key

What is 7.5% of 37?

Press	Display	Comments
7.5 %	0.075	
× 37 =	2.775	

Another key that may be useful is the reciprocal key $1/x$.

Example 2 – Using the 1/x Key

Convert $\dfrac{1}{6}$ to its decimal equivalent.

Press	Display	Comments
6 ■ 1/x	1.666667	

Automatic Shut-off

If your calculator is left on for several minutes without being used, it will shut off automatically.

Mortgage Financing Techniques

The Basis of Interest Rate Calculations

Interest is, essentially, rent charged for the use of borrowed funds, i.e., the principal amount. A loan contract will specify that interest will be charged at the end of a specified time period; for example, interest might be charged at the end of each month that the borrower has had the use of the funds. These interest periods are referred to as "compounding" periods; if interest is charged monthly, the loan is said to have monthly compounding. The amount of interest charged at the end of a compounding period is some specified percentage of the amount of principal the borrower has had use of during the *entire compounding period*. The percentage is referred to as the periodic interest rate or the interest rate per compounding period. The amount of interest charged at the end of the compounding period is equal to the amount of principal outstanding during the compounding period multiplied by the interest rate expressed as a decimal.

Borrowers and lenders are concerned with the interest rate per compounding period, and how often these payment periods occur (or the length of the compounding periods). For example, if $1,000 is borrowed at 1.5% per compounding period, the borrower will pay more interest in a year if this 1.5% is charged monthly than if it is charged semi-annually (that is, monthly compounding rather than semi-annual compounding). When analyzing a financial arrangement, whether it is a credit card balance, a demand loan, or a mortgage, one must know both the interest rate per compounding period and the frequency of compounding. Borrowers and lenders may agree on any interest rate, frequency of compounding, or frequency of payment. However, in Canada, a provision of the Interest Act requires the rate of interest to be quoted in a mortgage contract with either annual or semi-annual compounding. This provision has resulted in semi-annual compounding becoming the industry rule for mortgages.[1]

The basic concept of valuation of financial instruments focuses upon the relationship between when interest must be paid and when principal must be repaid. In the case of *simple interest*, interest is paid or earned each period on the original principal amount only, but not on any interest charged or paid. However, with *compound interest*, the interest charged changes with the interest that accumulates over time. Compound interest means that interest is charged (or earned) on interest (as well as on the principal amount).

The essential difference between simple and compound interest is that simple interest is based on the principal amount only, whereas with compound interest, the interest charged changes with the interest that accumulates over time. Note that compound interest need not be actually paid to the lender on a periodic basis; unpaid interest may be added to the debt and itself earn interest (an accrual loan or investment).

Nominal and Periodic Interest Rates

The annual interest rate generally quoted for compound interest is referred to as the "nominal interest rate per annum". The nominal rate is represented mathematically as j_m:

$$j_m = i \times m$$

This equation can be restated to determine the periodic rate:

$$i = \frac{j_m}{m}$$

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[1]   The reason semi-annual compounding is quoted rather than annual compounding is because it results in interest rates which appear to be lower than those based on annual compounding.

where:

j_m = Nominal interest rate compounded m times per year

m = Number of compounding periods per annum

i = Interest rate per compounding period, or periodic interest rate

The nominal rate (j_m) is always expressed as a certain percentage per year compounded a specific number of times during the year (m). The periodic rate (i) is the rate per compounding period, such as the rate per day, per week, per month, or per half-year.

Illustration 1

Consider the nominal rate of 12% per annum, compounded semi-annually, not in advance.[2] This would be expressed as:

$$j_2 = 12\%$$

and $$i = \frac{j_2}{2} = \frac{12\%}{2} = 6\%$$

Solution

Thus, the statement that interest is 12% per annum, compounded semi-annually tells the analyst that there are two (m) compounding periods per annum and that interest is to be 6% ($i = j_m \div m$) per semi-annual compounding period.

This can be illustrated using a *time diagram*[3] as shown below:

	6%		6%	
0		1		2 semi-annual periods
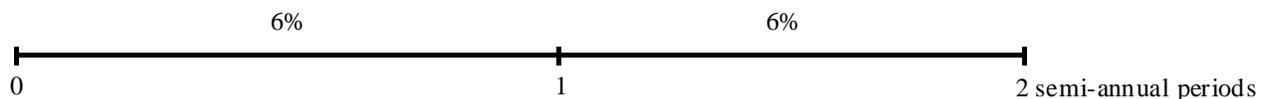
$j_2 = 12\%$ Nominal interest rate (per year)

m = 2   Number of periods per year

i = 6%   Periodic interest rate (semi-annual period)

Just as the nominal interest rate per annum (j_m) has an indicated frequency of compounding (m), it is also necessary to specify the frequency of compounding for periodic rates. The following shorthand notation is used in this supplement to indicate the frequency of compounding that is intended for periodic rates:

i_d represents an interest rate per daily compounding period

i_w represents an interest rate per weekly compounding period

i_{mo} represents an interest rate per monthly compounding period

i_q represents an interest rate per quarterly compounding period

i_{sa} represents an interest rate per semi-annual compounding period

i_a represents an interest rate per annual compounding period

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[2] "Not in advance" refers to the fact that the amount of interest accruing over the compounding period is calculated at the end of the compounding period, so that the borrower pays the interest at the end (or, not in advance) of the compounding period. Almost all rates of interest are calculated "not in advance". Therefore, the statement "not in advance" is frequently not used, and the interest rate would be quoted as 12% per annum, compounded semi-annually. Unless it is explicitly stated to be otherwise, students may assume that all interest rates are "not in advance".

[3] Time diagrams are shown as a horizontal line representing time. The present value is at the left (time 0) and the future value is at the right. In financial arrangements, time is measured by compounding periods, and so 2 semi-annual compounding periods are shown along the "time" line.

For example:

$$i_d = j_{365} \div 365$$
$$i_w = j_{52} \div 52$$
$$i_{mo} = j_{12} \div 12$$
$$i_q = j_4 \div 4$$
$$i_{sa} = j_2 \div 2$$
$$i_a = j_1 \div 1$$

Rearranging the above equations, we can also solve for all of the common nominal interest rates:

Daily compounding:	$j_{365} = i_d \times 365$
Weekly compounding:	$j_{52} = i_w \times 52$
Monthly compounding:	$j_{12} = i_{mo} \times 12$
Quarterly compounding:	$j_4 = i_q \times 4$
Semi-annual compounding:	$j_2 = i_{sa} \times 2$
Annual compounding:	$j_1 = i_a \times 1$

The interest rate j_1, which is the nominal rate per annum, compounded annually, is also known as the *effective annual interest rate*.

Completion of Exercise 1 should provide an increased familiarity with periodic interest rates, compounding frequency, nominal rates, and the interrelationship between them.

Exercise 1

The following tables represent a survey of interest rates quoted by financial institutions on term deposits. Complete the tables by entering the appropriate values for the question marks for either the periodic rate, the number of compounding periods, or the nominal rate.

Question	Periodic Rate i	Number of Compounding Periods per Year (m)	Nominal Rate $(j_m = i \times m)$
SAMPLE	3.5%	2	$j_2 = 7\%$
(a)	1.0625%	12	$j_{12} = ?$
(b)	2.275%	4	$j_4 = ?$
(c)	0.035616%	m = ?	$j_m = 13\%$

Question	Nominal Rate j_m	Number of Compounding Periods per Year (m)	Periodic Rate $(i = j_m \div m)$
SAMPLE	$j_2 = 10\%$	2	$i_{sa} = 5\%$
(d)	$j_{12} = 6\%$	12	$i_{mo} = ?$
(e)	$j_{365} = 8\%$	365	$i_d = ?$
(f)	$j_m = 15\%$	m = ?	$i_{mo} = 1.25\%$

Solution

(a)	$j_{12} = 12.75\%$		(d)	$i_{mo} = 0.5\%$
(b)	$j_4 = 9.1\%$		(e)	$i_d = 0.0219178\%$
(c)	$m = 365$		(f)	$m = 12$

Compound Interest Calculations

As an introduction to the nature of compound interest calculations, consider Illustration 2:

Illustration 2

A commercial enterprise has arranged for an interest accrual loan where the $10,000 amount borrowed is to be repaid in full at the end of three years. The borrower has agreed, in addition, to pay interest at the rate of 15% per annum, compounded annually on the borrowed funds. Calculate the amount owing at the end of the 3-year term of the loan.

Solution

Given that the borrower owes $10,000 throughout the year, the amount of interest owing at the end of the one-year term is calculated as follows:

Interest Owing (I) = Principal borrowed × interest rate per interest calculation period
(in this example, interest is calculated per *annual* compounding period)
Interest Owing (I) = $10,000 × 15%
Interest Owing (I) = $10,000 × 0.15
Interest Owing (I) = $1,500

Thus, the amount of interest owing at the end of the one-year term is $1,500. The total amount owing at the end of the start of the second year of this interest accrual loan would be the principal borrowed ($10,000) plus the interest charged ($1,500) or $11,500. The outstanding balance at the end of the third year can be calculated manually:

Amount owing at end of year one = $10,000 × 1.15 = $11,500
Amount owing at end of year two = $11,500 × 1.15 = $13,225
Amount owing at end of year three = $13,225 × 1.15 = $15,208.75

A simpler calculation recognizes that for each annual compounding period the principal outstanding is multiplied by 1.15:

FV = $10,000 × 1.15 × 1.15 × 1.15
FV = $15,208.75

Or, this equation can be expressed in exponential form:

$$FV = \$10,000 \times (1.15)^3$$

In more general terms, the future value formula is as follows:

$$FV = PV \times (1 + i)^n$$

where   FV = Future value (or amount owing in the future)
PV = Present value (or original amount borrowed)
i = interest rate per compounding period expressed as a decimal
n = number of compounding periods in the loan term

This illustration introduces a number of very important definitions and concepts. Financial analysts use short form abbreviations for the loan amount, interest rates, and other mortgage items. In this shorthand notation, the following symbols are used:

PV = Present value: the amount of principal owing at the beginning of an interest calculation period

FV = Future value: the amount of money owing in the future

i = Interest rate per compounding period; the fraction (or percentage) used to calculate the dollar amount of interest owing

I = Interest owing, in dollars, at the end of an interest calculation (compounding) period

n = Number of compounding periods contracted for

The HP 10bII+ calculator is pre-programmed for exponential calculations. The steps below show how the calculator can be used to determine the amount owing on the loan by using the exponential function.

Press	Display	Comments
1.15 ■ y^x 3 =	1.520875	1.15 raised to the power of 3
× 10000 =	15,208.75	Total amount owed at end of year 3

Reminder

For illustrations in this course, the HP 10bII+ calculator is set to six decimal places by pressing ■ DISP 6. However, for ease of presentation, on the display portion of calculator steps, we will not show the display with zeros when they do not impact the result (and are mathematically insignificant).

When using the HP 10bII+ calculator, the above formula, $FV = PV \times (1 + i)^n$, must be slightly modified to consider nominal interest rates. Recall that a periodic rate is equal to the nominal rate divided by the compounding frequency. Thus, the formula becomes:

$$FV = PV \times (1 + j_m/m)^n$$

where j_m = Nominal interest rate per annum

m = Compounding frequency

n = Number of compounding periods in the loan term

This modified version is necessary because this calculator only works with nominal interest rates. This formula for interest accrual loans has been pre-programmed into the mortgage finance keys of the HP 10bII+ calculator. These keys are:

I/YR   Nominal interest rate per year (j_m) – entered as a percent amount (not as a decimal)

■ P/YR   "Periods per year" (m) – this indicates the compounding frequency of the nominal rate in I/YR and is located on the lower half of the PMT key

N   Number of compounding or payment periods in the financial problem – this number will be expressed in the same frequency as P/YR (in other words, if P/YR is 12, then N will represent the number of months)

PV   Present value

FV   Future value after N periods

PMT   Payment per period – this is expressed in the same frequency as P/YR and N, i.e., if N is months, PMT represents the payment per month

Positive and Negative Cash Flows

The internal operation of the HP 10bII+ calculator requires that all financial calculations have at least one positive and one negative cash flow. This means that at least one of the PV, FV, and PMT keys will have to be shown and/or entered as a negative amount. Generally, cash flowing in is positive, while cash flowing out is negative.

For example, in mortgage loan problems, the borrower receives loan funds at the beginning of the term (cash in, so a positive amount) and pays back the loan funds either during or at the end of the term (cash out, so negative amounts). In this type of problem, PV will be shown/entered as a positive, while PMT and FV will be shown/entered as negatives. When entering a negative amount, the +/– key is used, not the – key.

Similarly, from an investor's perspective, the initial investment is paid out (cash out, so negative amount) and the investor receives money in the future (cash in, so positive amount). In this type of problem, PV will be shown/entered as a negative, while PMT and FV will be shown/entered as positives.

Summary

Borrower's Perspective   Investor's Perspective
PV   +   PV   –
PMT   –   PMT   +
FV   –   FV   +

The internal operation of the HP 10bII+ calculator requires that all financial calculations have at least one positive and one negative cash flow. This means that at least one of the PV, FV, and PMT keys will have to be shown as a negative amount. Generally, money invested is shown as a negative amount and money withdrawn is shown as positive. In other words, cash flowing out is negative, while cash flowing in is positive. For example, from a borrower's perspective, cash borrowed is a positive amount and cash paid back is negative. Similarly, from an investor's perspective, the initial investment is negative and the money received in the future is a positive amount. This distinction will be made clearer by the following example.

PV = $10,000   PMT = 0   i = 15%

0 ▲   1   2   3 years

                                                              ▼
                                                           FV = ?
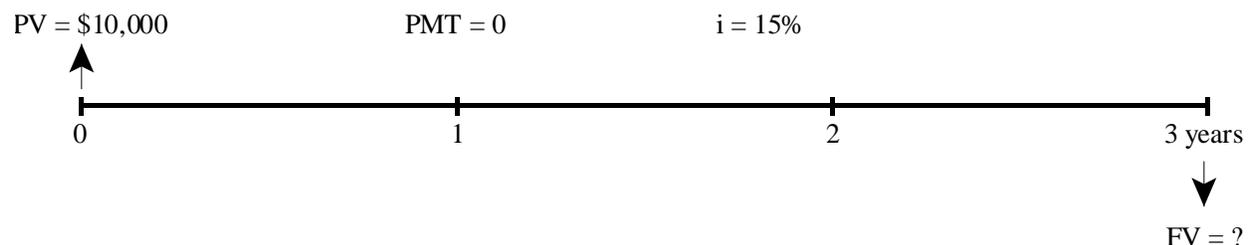
Illustration 2 is illustrated above with a time diagram, but with a new feature added. The cash flows are placed along the horizontal line with an arrow representing positive or negative cash flows. An "up arrow" represents a positive cash flow (money received), while a "down arrow" represents a negative cash flow (money paid out).

To solve this problem with the HP 10bII+ calculator, a number should be entered and then "labelled" appropriately. For example, the loan in this example has a 3-year term, so "3" should be entered and then "N" pressed in order to enter a value of 3 as the number of compounding periods during the term. By entering a number and then labelling it, you can *enter the information in any order*.

Press	Display	Comments
15 I/YR	15	Enter nominal interest rate per year
1 ■ P/YR	1	Enter compounding frequency
3 N	3	Enter number of compounding periods
0 PMT	0	No payments during term
10000 PV	10,000	Enter present value (the borrower receives the cash, so it is entered as a positive amount)
FV	−15,208.75	Computed future value (this will be paid out by the borrower, so it is a negative amount)

This is the same answer as that calculated with either of the two approaches shown earlier, but with much less work needed.

Helpful Hint: Using the RCL Key

Note that if you enter an incorrect number on the screen, it can be cleared by pushing C once. If you enter an incorrect number into any of the six financial keys, N, I/YR, PMT, PV, FV, P/YR, it can be corrected by re-entering the desired number into that key. You can verify what information is stored in each of the above financial keys by pressing RCL and then the corresponding financial key you are interested in. For example, if you obtained an incorrect solution for the example above, you can check what is stored in N by pressing RCL N; I/YR by pressing RCL I/YR, etc.

Future Value Calculations in Excel

You can solve future value of lump sum calculations using Excel by either using the math formula or the financial functions. With the FV function, you enter the periodic rate, the loan or investment period (NPER), payment (0), PV (the loan amount), and type of loan (0). The type of loan is a value representing the timing of the payment. If payments occur at the beginning of a period, the type is 1 and if the payments occur at the end of a period, the type is 0. Since there is no payment in this calculation, the type is 0 or can be omitted as 0 is the default option.

Notice that the result shows as a negative, like the calculator solution.

Note also that the compounding frequency of the interest rate must match the periods used in specifying the loan or investment period (NPER) – if not, an interest rate conversion is required.

Equivalent Interest Rates

The basis upon which interest rate calculations are performed is stated as follows:

Two interest rates are said to be equivalent if, for the same amount borrowed, over the same period of time, the same amount is owed at the end of the period of time.

One particular equivalent interest rate, the equivalent rate with annual compounding (j_1), is called the effective annual rate. By convention, the effective rate is used to standardize interest rates to allow borrowers and lenders to compare different rates on a common basis. The financial calculator also uses the effective annual interest rate to convert between equivalent nominal interest rates.

Illustration 3

Assume that a bank agrees to give a loan at an interest rate of 14% per annum, compounded monthly. In order to determine the rate that the bank must disclose under the Interest Act, calculate the nominal rate per annum with semi-annual compounding that is equivalent to $j_{12} = 14\%$.

Solution

This problem can be solved using mathematical formulas, but this involves complex and time-consuming algebra. The alternative, and equally valid, approach to calculate equivalent interest rates is to use the financial keys of a business calculator. The process involves entering the nominal interest rate provided and converting it to its effective annual equivalent. Then, the desired compounding frequency is entered, which is usually the number of payment periods per year. The final step is to solve for the equivalent nominal rate with the desired compounding frequency. It is important to note that the HP 10bII+ works with nominal interest rates in the financial keys (some financial calculators work with periodic interest rates). To solve for a periodic rate, one must divide the nominal rate by its compounding frequency.

There are two other financial keys of the HP 10bII+ which have not yet been introduced, but are needed for interest rate conversion problems. These are:

- ■ NOM%   Nominal interest rate per year (j)
- ■ EFF%   Effective interest rate (j_1) which is calculated based on the nominal rate (j) in NOM% and the compounding frequency (m) entered in P/YR

Enter the given nominal rate and the stated number of compounding periods per year (12, in this case). Solve for the effective annual rate (the nominal rate with annual compounding). Then, enter the desired compounding periods (2, in this case). Solve for the equivalent nominal rate.

Press	Display	Comments
14 ■ NOM%	14	Enter stated nominal rate[4]
12 ■ P/YR	12	Enter stated compounding frequency
■ EFF%	14.934203	Compute effective annual interest rate
2 ■ P/YR	2	Enter desired compounding frequency
■ NOM%	14.414741	Compute equivalent j_2 rate

The nominal rate per annum with semi-annual compounding equivalent to $j_{12} = 14\%$ is $j_2 = 14.414741\%$. If it were necessary to calculate the periodic rate per semi-annual period, this could be done by dividing the nominal rate of $j_2 = 14.414741\%$ by the number of compounding periods per year (2) to obtain the periodic rate of $i_{sa} = 7.207371\%$.

Illustration 4

Assume that a bank agrees to give a loan at an interest rate of 9% per annum, compounded semi-annually. Calculate the equivalent nominal rate per annum with monthly compounding (j_{12}).

Solution

Enter the given nominal rate and the stated number of compounding periods per year (2, in this case). Solve for the effective annual rate (the nominal rate with annual compounding). Then, enter the desired compounding periods (12, in this case). Solve for the equivalent nominal rate.

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[4]   In the interest rate conversions illustrated in this supplement, the first step shown is to enter the stated nominal rate using ■ NOM%. Students may notice that similar results can also be achieved by pressing I/YR alone.

Press	Display	Comments
9 ■ NOM%	9	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	9.2025	Compute effective annual interest rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	8.835748	Compute equivalent j_{12} rate

The nominal rate per annum with monthly compounding equivalent to $j_2 = 9\%$ is $j_{12} = 8.835748\%$. If it were necessary to calculate the monthly periodic rate, this could be done by dividing the nominal rate of $j_{12} = 8.835748\%$ by the number of compounding periods per year (12) to get the periodic rate of $i_{mo} = 0.736312\%$.

Exercise 2

Consider the following table:

1. The first column specifies a nominal rate of interest with a given compounding frequency.
2. The second column provides the desired compounding frequency.
3. The third column presents an equivalent nominal interest rate with the desired frequency of compounding.

You should ensure that you are able to use the nominal rates of interest and desired frequencies of compounding shown in the first two columns to calculate the equivalent nominal interest rate given in the third column. This skill is critical to completing all of the following mortgage finance calculations.

Nominal interest rate	Desired number of compounding periods per annum	Equivalent nominal interest rate with desired compounding frequency
$j_{12} = 12\%$	1	$j_1 = 12.682503\%$
$j_2 = 10\%$	12	$j_{12} = 9.797815\%$
$j_4 = 8\%$	2	$j_2 = 8.08\%$
$j_1 = 9\%$	365	$j_{365} = 8.618787\%$
$j_4 = 7.5\%$	12	$j_{12} = 7.453607\%$
$j_1 = 6\%$	12	$j_{12} = 5.841061\%$

Excel Tip: Interest Rate Conversions

You can solve for equivalent interest rates in Excel by programming in the mathematical formulas:
Periodic Rate= ((1+(Nominal Rate/Stated Frequency))^(Stated Frequency/Desired Frequency))-1

The syntax for this formula is difficult and prone to errors. An alternative and much simpler solution in Excel is to use Excel's functions, NOMINAL and EFFECT. These operate in much the same way as the NOM% and EFF% keys on the HP 10bII+.

= NOMINAL (effective rate per year, N compounding periods per year)
= EFFECT (nominal rate per year, N compounding periods per year)

Future Value and Present Value for Interest Accrual Loans/Investments

One type of compound interest calculation that is frequently encountered relates to the future and present values of single payment or lump sum amounts (interest accrual loans or investments).

Calculation of Future Value

This calculation was explained and illustrated earlier:

$$FV = PV(1 + i)^n$$

Illustration 5

Assume you arrange an investment of $20,000 yielding interest at 11% per annum, compounded annually. What is the future value of this investment after 15 months?

Solution

$$FV = PV(1 + i)^n$$

where

PV = $20,000
j_1 = 11% (11% per annum, compounded annually)
n = 15 months
FV = ?

In the absence of information, it is assumed that payments are zero. Note that in this illustration, "n" is expressed in months and the interest rate is compounded annually. Therefore, the first step is to find the equivalent nominal rate, compounded monthly (j_{12}).

Press	Display	Comments
11 ■ NOM%	11	Stated nominal rate
1 ■ P/YR	1	Stated compounding frequency
■ EFF%	11	Equivalent effective annual rate
12 ■ P/YR	12	Desired compounding frequency
■ NOM%	10.481513	Equivalent j_{12} rate
15 N	15	Number of months
20000 +/– PV	−20,000	Amount of investment
0 PMT	0	No payments
FV	22,786.819865	Future value

The future value of this investment after 15 months would be $22,786.82.[5]

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[5]   Students may notice that this problem can be solved without needing to do this interest rate conversion. If the 15 months are entered into N as 1.25 years and the P/YR is entered as 1, then the I/YR can be entered as 11 and the same future value will result.

Rounding Rules

When calculating monetary amounts, numbers will have to be rounded off, since it is impossible to pay or receive an amount less than one cent. When rounding monetary values (e.g. present value, future value, or payment), normal rounding rules are applied. This is the common mathematical rule that states:

- If the third decimal is 5 or greater, the number is rounded up: e.g. 8,955.436 would be rounded UP to $8,955.44 (because the third decimal is a 6).
- If the third decimal is less than 5, the number is rounded down: e.g. 8,955.433 would be rounded DOWN to $8,955.43 (because the third decimal is a 3).

In this course, assume all monetary values are rounded to the nearest cent, *unless instructed otherwise*.

Calculation of Present Value

In order to calculate the present value of a single future value, we need to rearrange the basic relationship between the present value and a future value of a lump sum, based on compound interest. This can be expressed as follows:

$$PV = FV(1 + i)^{-n}$$

This is the normal expression for calculating the present value of a future lump sum. However, with modern calculators we need only be concerned about entering the known data and computing the unknown value.

Illustration 6

You are offered an investment that will produce $350,000 in 10 years. If you wish to earn 9% compounded semi-annually, how much should you offer to pay for the investment today?

Solution

$$PV = FV (1 + i)^{-n}$$

where:

FV = $350,000
j_2 = 9% (9% per annum, compounded semi-annually)
n = 10 years

The solution requires you to calculate the present value based on the desired yield. In this case, the investment term is expressed in years (n) and the interest rate is compounded semi-annually. To solve, we need to calculate the equivalent nominal rate, compounded annually.

Press	Display	Comments
9 ■ NOM%	9	Stated nominal rate
2 ■ P/YR	2	Stated compounding frequency
■ EFF%	9.2025	Equivalent effective annual rate
1 ■ P/YR	1	Desired compounding frequency
■ NOM%	9.2025	Equivalent j_1 rate
350000 FV	350,000	Expected future value
10 N	10	Number of years
0 PMT	0	No payments
PV	−145,125.00089	Present value

You should offer $145,125 for the investment today.[6]

Annuity or Payment Calculations

Up to this point we have been doing calculations involving only one-time lump sum cash flows. In order to do calculations involving recurring payments, we can use the PMT key on the calculator. In order to use the PMT key, payments must be in the form of an annuity. An annuity is a stream of equal payments, which are spread evenly over time. An example of an annuity is the stream of payments of a constant payment mortgage, which is the most common application of the PMT key. Another example of an annuity would be monthly deposits to a bank account to accumulate some amount in the future.

Recurring Payments

Illustration 7

An individual would like to put aside some money into a savings account to accumulate money to buy a boat. If she can put aside $200 at the end of every month, and the savings account earns interest at $j_{12} = 6\%$, how much money will have accumulated in the savings account by the end of the fourth year?

Solution

In order to calculate the amount in the savings account at the end of the fourth year, we must enter the information into the financial keys of the calculator. As before, we must enter information in all but one of the financial keys in order to calculate the final piece of information. The information given in the problem is as follows:

$$PV \quad = \quad 0$$
$$N \quad = \quad 48 \text{ compounding periods } (4 \times 12 = 48)$$
$$j_{12} \quad = \quad 6\%$$
$$PMT \quad = \quad \$200 \text{ (paid out, so they will be negative amounts)}$$
$$FV \quad = \quad ? \quad \text{(cash received, so it will be a positive amount)}$$

---

[6]   As in the previous illustration, this problem can be solved without needing to do an interest rate conversion. If the 10 years are entered into N as 20 semi-annual periods and the P/YR is entered as 2, then the I/YR can be entered as 9 and the same present value will result.

FV = ?

PV = 0
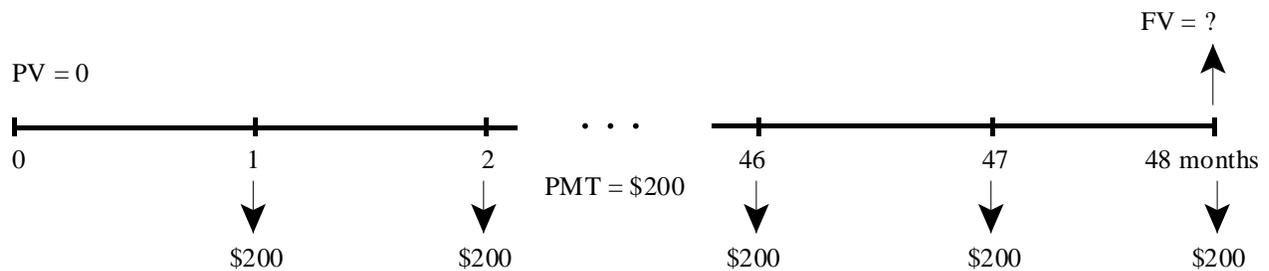
As the frequencies of payment and compounding correspond (both are monthly), this problem may be solved without an interest rate conversion.

Press	Display	Comments
6 I/YR	6	Nominal interest rate
12 ■ P/YR	12	Compounding frequency
48 N	48	Number of payments
0 PV	0	No money in the account at beginning
200 +/– PMT	−200	Payment
FV	10,819.566444	Future value

By depositing $200 into a savings account at the end of each month for 48 months, the individual will accumulate $10,819.57 at the end of four years (48 payments).

A stream of cash flows, such as the one in the above illustration, where regular payments are being set aside to accumulate money for some specific purpose in the future is known as a "sinking fund". Sinking funds are often used by businesses to accumulate money to repay a bond, or to replace worn machinery or equipment.

Frequency of Payments and Compounding Period

Notice that in the previous calculation, the frequency of compounding of the interest rate and the frequency of the payments matched. When using the financial keys, and the PMT key in particular, it is vital that the I/YR, N, and PMT keys all use the same frequency. For example, if payments were made semi-annually, the interest rate would have to be entered in the calculator as a j_2 rate (I/YR is a j_2, P/YR is 2), N would be the number of semi-annual payments, and PMT would be the amount of the semi-annual payments.

Excel Tip: Present and Future Value

Excel's equivalent functions for calculating present value and future value:

- PV calculates a present value of an investment given the interest rate per period, the number of payment periods, the payments per period (can be zero), the future value at the end of the term (can be zero) and the type of annuity (0 = payments at end of period; 1= payments at beginning of period).
      = PV(rate,nper,pmt,fv,type)

- FV calculates a future value of an investment given the interest rate per period, the number of payment periods, the payments per period (can be zero), the present value at the start of the term (can be zero) and the type of annuity (0 = payments at end of period; 1= payments at beginning of period).
      = FV(rate,nper,pmt,pv,type)

Tips:

- The formulas have to consider positive and negative cash flows: funds in are a positive, funds out are a negative.
- For PV, if PMT is zero, then an FV must be specified; or, if FV is zero, then a PMT must be specified.
- For FV, if PMT is zero, then a PV must be specified; or, if PV is zero, then a PMT must be specified.
- In the Excel cell where the formula is stored, click on the PV or FV and it will show you the formula, then click on the inputs and it will highlight which input you have specified, along with colour coding them on your spreadsheet – this helps identify where you have made errors.
- Double click on the PV or FV in the formula and it will open up the Excel Help entry for that formula, explaining each term and giving examples of how to specify the formula – very helpful resource!

Calculations for Constant Payment Mortgages

The HP 10bII+ financial calculator is pre-programmed to calculate loan amounts (PV), future values (FV), payments (PMT), amortization periods (N), and interest rates (I/YR). By entering any four of these variables (PV, FV, PMT, N, and I/YR), the calculator can then determine the fifth variable.

The following conditions must occur in order to use the calculator to analyze a constant payment mortgage:

1. The present value must occur at the *beginning* of the first payment/compounding period.

2. The payments must be equal in amount, occur at regular intervals, and be made at the end of each payment period.

3. The rate of interest must be stated as, or converted to, a nominal rate with compounding frequency matching the payment frequency.

Illustration 8

A local trust company has been approached by a real estate investor desiring mortgage money. The investor will pay $4,000 per month over a 15-year period. What size of loan will the trust company advance if it desires a yield (or interest rate) of $j_2 = 4\%$?

Solution

When a financial arrangement has a different frequency of compounding and payment, it is necessary to convert the given nominal rate of interest with the stated compounding frequency to an equivalent nominal interest rate for which the compounding frequency matches the payment frequency. In the above illustration, the lender demands a return on investment of 4% per annum, compounded semi-annually. The borrower, on the other hand, is making payments on a monthly basis.

The first step to solve for the maximum allowable loan amount involves calculating the nominal rate of interest with monthly compounding (j_{12}) that is equivalent to $j_2 = 4\%$.

Press	Display	Comments
4 ■ NOM%	4	Stated nominal rate
2 ■ P/YR	2	Stated compounding frequency
■ EFF%	4.04	Equivalent effective annual rate
12 ■ P/YR	12	Desired compounding frequency
■ NOM%	3.967068	Equivalent j_{12} rate

The borrower will make 180 monthly payments (15 years $\times$ 12 payments per year) of $4,000, and the rate of interest is 3.967068% per annum, compounded monthly. Since the rate of $j_{12} = 3.967068\%$ is already entered as the nominal interest rate with monthly compounding, it does not have to be entered again. Equivalent interest rates should *not* be "keyed" into the calculator. Instead, they should be calculated and used directly to avoid errors in re-entering the number.

After determining the nominal rate, the maximum loan amount would be calculated as follows:

(continued)		
Press	**Display**	**Comments**
	3.967068	j_{12} rate displayed from previous calculation
4000 +/– PMT	–4,000	Payment per month
15 × 12 = N	180	Number of monthly payments
0 FV	0	Indicates that FV is not to be used (because all of the loan is totally repaid at the end of 180 months)
PV	541,976.976891	Present value or loan amount

The lender, desiring to earn 4% per annum, compounded semi-annually, would be willing to advance $541,976.98 in exchange for the borrower's promise to pay $4,000 per month for 180 months.

Illustration 9(a)

An individual is thinking of buying a residential condominium but wants to limit mortgage payments to $700 per month. If mortgage rates are 12% per annum, compounded monthly, and the lender will permit monthly payments to be made over a 25-year amortization period, determine the maximum allowable loan.

Solution

The financial terms of the proposed loan may be summarized as follows:

$$PMT = \$700 \text{ per month}$$
$$N = 25 \times 12 = 300 \text{ months}$$
$$j_{12} = 12\%$$

PV = ? $j_{12} = 12\%$ FV = $0
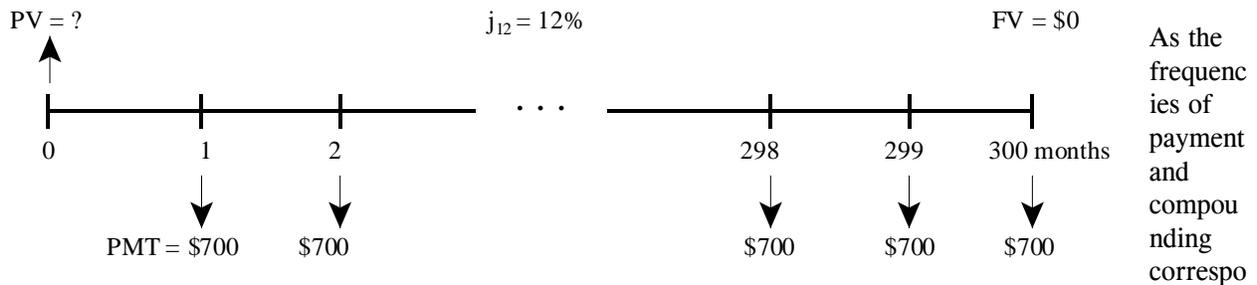
As the frequencies of payment and compounding correspond (both are monthly), the problem may be solved directly:

Press	Display	Comments
12 I/YR	12	Nominal rate
12 ■ P/YR	12	Compounding frequency
700 +/− PMT	−700	Monthly payment[7]
25 × 12 = N	300	Amortization period in months
0 FV	0	This calculation will not use a future value amount so zero must be entered[8]
PV	66,462.585878	Present value (or loan amount)

The maximum loan based on the interest rate, payments and amortization period specified, is $66,462.59.

Illustration 9(b)

If the loan above called for interest at the rate of 15% per annum, compounded monthly, determine the maximum loan amount.

Solution

N = 300; j_{12} = 15%; PMT = $700; PV = ?

Because PMT, N, P/YR, and FV are already stored and do not require revision, the calculation is:

(continued)		
Press	Display	Comments
15 I/YR	15	Stated nominal rate
PV	54,652.035481	Loan amount

---

[7] Most of the calculations in the remainder of this supplement are for mortgage loans. In these problems the borrower receives loan funds at the beginning of the loan term (cash in, so a positive amount) and makes periodic payments during the loan term and an outstanding balance payment at the end of the loan term (cash out, so negative amounts). In these examples, PV will be shown as positive, while PMT and FV will be shown as negatives.

[8] A future value amount is not used in this problem because at the end of 300 months the entire principal amount (or outstanding balance) has been repaid, making the future value of the loan zero.

Thus, increasing the interest rate from $j_{12} = 12\%$ to $j_{12} = 15\%$ has the effect of decreasing the maximum allowable loan by almost $12,000 (from $66,462.59 to $54,652.04).

From the preceding examples, it is clear that the rate of interest charged on a loan can have a large impact on the size of the loan a *fixed* series of payments will support. With constant payment mortgage loans, a large portion of each of the early payments is allocated to the payment of interest. Increased interest rates reduce the amount of each payment available for principal repayment, making a very large impact on an individual's ability to borrow a given amount. These examples assume that there are no other borrower qualifications, which is not typically the case in mortgage lending.

Illustration 10

A mortgage loan for $60,000 is to be repaid by equal monthly payments over a 30-year period. The interest rate is 5% per annum, compounded monthly. Calculate the size of the required monthly payments.

Solution

$j_{12} = 5\%$; N = 12 × 30 = 360; PV = $60,000; PMT = ?

Press	Display	Comments
5 I/YR	5	Stated nominal rate
12 ■ P/YR	12	Stated compounding frequency
30 × 12 = N	360	Amortization period in months
60000 PV	60,000	Present value
0 FV	0	Loan is fully repaid at the end of 360 months
PMT	-322.092974	Payment
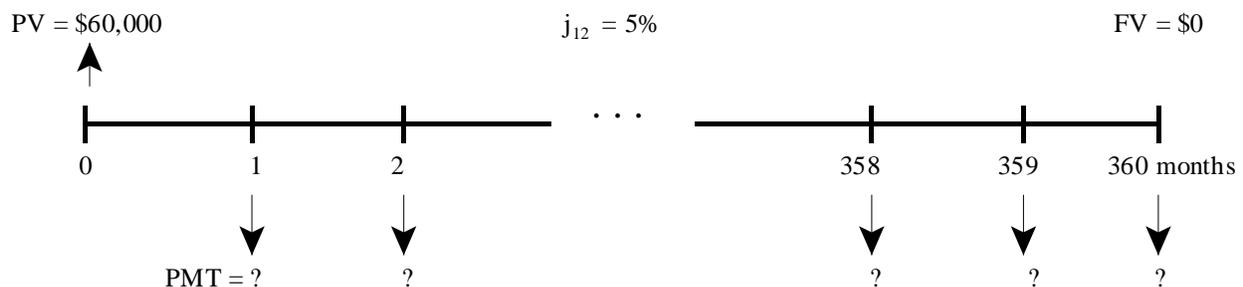
The calculated monthly payments are $322.092974. Since borrowers cannot make payments that involve fractions of cents, the payments must be rounded to at least the nearest cent. Regular rounding rules apply unless the facts indicate otherwise (e.g., an example may ask that payments be rounded up to the next higher $10 or $100 to obtain a round number). Therefore, the payments on this loan would be $322.09.

Excel Tip: Payments

Excel's equivalent functions for calculating payments:

- PMT calculates a payment given the interest rate per period, the number of payment periods, the present value, the future value (OSB at end of term), and the type of annuity (0 = payments at end of period; 1 = payments at beginning of period).

  = PMT (rate,nper,pv,[fv],[type])

Calculation of Outstanding Balances

It is important to know how to calculate the outstanding balance of a loan, or the amount of principal owing at a specific point in time, for several reasons. Most vendors want to know how much they will receive from the sale of their property after they have repaid the outstanding balance on their mortgage. While mortgage payments are calculated using the amortization period, the actual length of the mortgage contract may be different from the amortization period. The length of the mortgage contract is called the term. If the mortgage term and amortization period are the same length of time, the mortgage is said to be fully amortized. If the mortgage term is shorter than the amortization period, the mortgage is said to be partially amortized. Since mortgages are typically partially amortized with one to five-year contractual terms, the amount of money which the borrower owes the lender when the contract expires must be calculated.

To calculate the outstanding balance on an amortized loan, the payments are first calculated based on the full amortization period:

PV

FV = $0

0

Amortization
Period

PMT   PMT   PMT   PMT   PMT   PMT   PMT   PMT   PMT
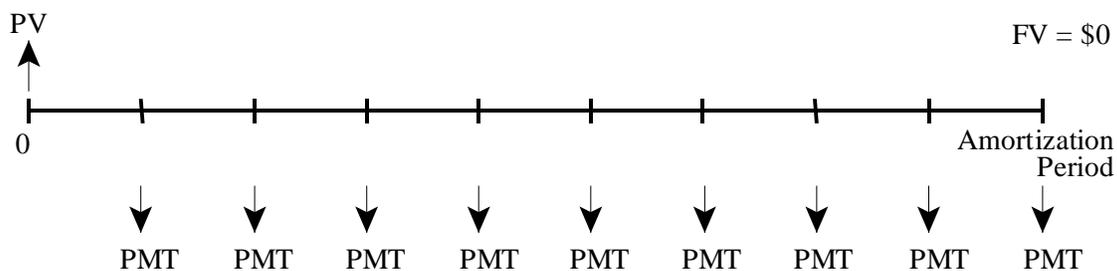
The outstanding balance is then calculated at the end of the loan term:

PV

No further payments made

0

Term

PMT   PMT   PMT   PMT   PMT

FV (OSB)
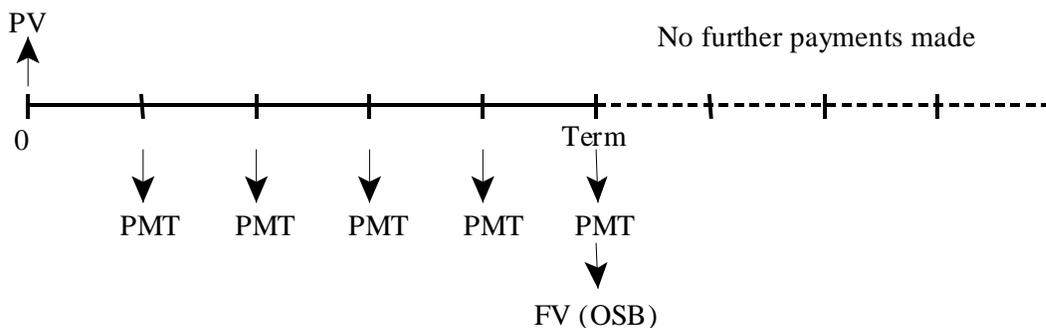
As shown below, the outstanding balance can be calculated quickly on your calculator.

Illustration 11

A $60,000 mortgage loan, written at $j_{12} = 12\%$, has a 20-year amortization, a 3-year term, and monthly payments, rounded up to the next higher dollar. What is the outstanding balance of the mortgage at the end of its term? In other words, what is the outstanding balance just after the 36th payment (OSB36) has been made?

Solution

(a)   *Calculate the size of the required monthly payments:*

   PV = $60,000
   N = 20 × 12 = 240 months
   j_{12} = 12%
   PMT = ?

PV = $60,000 $j_{12} = 12\%$ FV = $0
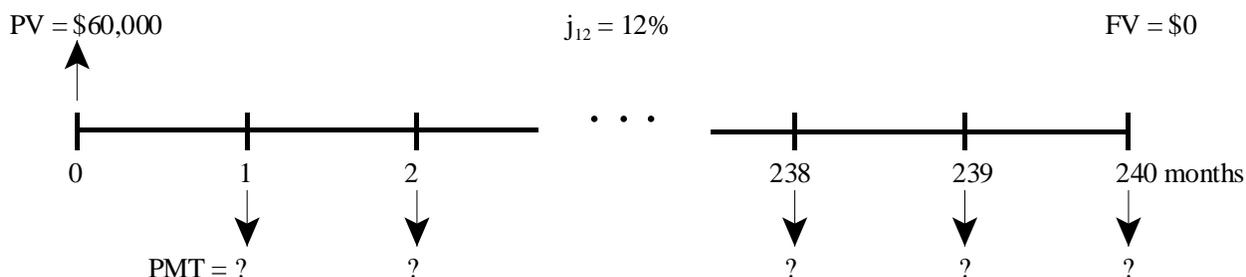
Press	Display	Comments
12 I/YR	12	Stated nominal rate
12 ■ P/YR	12	Stated compounding frequency
60000 PV	60,000	Loan amount
20 × 12 = N	240	Amortization period
0 FV	0	Payment is calculated to fully amortize loan
PMT	−660.65168	Payment

The monthly payment is $661, rounded up to the next higher dollar.

(b)   *Calculate the outstanding balance due immediately after (with) the 36th monthly payment:*

PV = $60,000 $j_{12} = 12\%$ End of Term
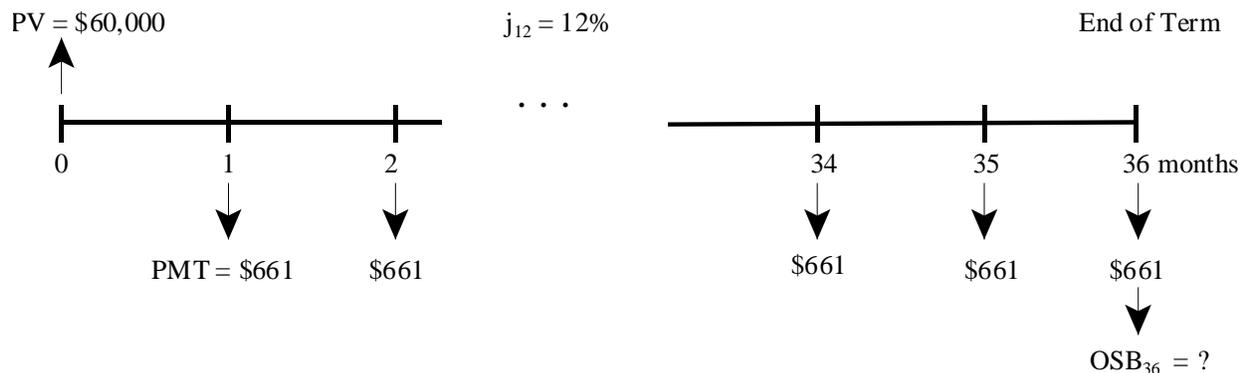
The HP 10bII+ calculator has a pre-programmed function that calculates outstanding balances. Before the outstanding balance can be calculated, the rounded payment must be entered into the calculator. The $661 payment is slightly higher than the $660.65168 payment required to repay the loan as calculated using the initial

information. Since the loan amount and interest rate are set by contract, the increase in the size of the payment causes more principal to be repaid in each month than is required to amortize the loan. This results in a faster repayment of the loan amount and, consequently, reduces the number of full payments needed to amortize the loan. When the new (larger) payment is entered, the pre-programmed function of your calculator revises the amortization period as part of the outstanding balance calculation:

(continued)		
Press	**Display**	**Comments**
	−660.65168	Payment from previous calculation
661 +/− PMT	−661	Rounded payment
N	239.477462	Recompute amortization period given higher payment*
36 INPUT ■ AMORT	36–36	PER 36-36**
= = =	57,372.310422	OSB$_{36}$

*This step is included for illustrative purposes and is not required in outstanding balance calculations. Recomputing the amortization period will not affect further calculations such as the outstanding balance, as long as the rounded payment has been re-entered.
**If you have the HP 10bII+ set to use *floating* decimal notation, the calculator will generate a "running" error when using the AMORT and INPUT keys. You can solve this problem by setting the decimals to a specific number. For example, if you press ■ DISP 6, this will set the calculator to 6 decimal places. This solves the running error and provides sufficient accuracy for this course.

If this loan was fully amortized, it would take 239.477462 months to pay off the loan: 239 payments of $661 plus a final 240th payment, which is less than $661. However, this loan is partially amortized and the borrower will instead make 36 monthly payments of $661 and pay the remaining balance of $57,372.31 at the end of the 36th month.

You may have noticed that while pressing the "=" sign three times, three different numbers appeared on the screen. The first number that appears is the principal paid in the 36th payment, the second number is the interest paid in the 36th payment, and the final number is the outstanding balance owing immediately after the 36th payment. These functions will be explained further in the next section.

Excel Tip: Outstanding Balances

Excel's equivalent functions for calculating outstanding balances:

- = FV(rate,nper,pmt,pv,type) – Calculates the future value for a loan based on constant payments and a constant interest rate.

  rate – periodic interest rate for the loan

  nper – term of the loan

  pmt – payment made each period

  pv – present value

  type – 0 indicates payments are due at the end of each period; 1 indicates payments are due at the beginning of each period

To determine the outstanding balance accurately, you must calculate it based on the rounded payment.

Calculation of Principal and Interest Components of Payments

In addition to the outstanding balance, it is often necessary to calculate the principal and interest components of payments on constant payment mortgages. These calculations are important because interest on payments can sometimes be deducted as an expense for income tax purposes. As well, borrowers like to know how much principal they have paid off in a single payment or over a series of payments. The calculation of principal and interest components of payments is done using the same keys on the calculator as the outstanding balance calculation shown above. This function will be explained using the following illustration.

Illustration 12

Three years ago, Tom and Nancy bought a house with a mortgage loan of $175,000, written at $j_2 = 9.5\%$, with a 25-year amortization, a 3-year term, and monthly payments rounded up to the next higher dollar. Tom and Nancy are about to make their 36th monthly payment, the last one in the loan's term, and want to know the following information:

(a) *How much principal will they be paying off with their 36th payment?*
(b) *How much interest will they be paying with their 36th payment?*
(c) *What will be the amount they will have to refinance after the 36th payment, i.e., the amount they will have to refinance after the 36th payment?*
(d) *How much principal did they pay off during the 3-year term?*
(e) *How much interest did they pay over the entire 3-year term?*

Solution

In order to answer any of these questions, it is necessary to first find the j_{12} interest rate and the monthly payments under the mortgage.

Press	Display	Comments
9.5 ■ NOM%	9.5	Stated nominal rate
2 ■ P/YR	2	Stated compounding frequency
■ EFF%	9.725625	Equivalent effective annual rate
12 ■ P/YR	12	Desired compounding frequency
■ NOM%	9.31726	Equivalent nominal rate with monthly compounding
175000 PV	175,000	Loan amount
25 × 12 = N	300	Amortization period
0 FV	0	Payment is calculated to fully amortize loan
PMT	−1,506.798355	Unrounded monthly payment
1507 +/− PMT	−1,507	Rounded payment

The monthly payment on the loan will is $1,507, rounded up to the next higher dollar. The answers to questions (a), (b), and (c) can be found using the calculator's pre-programmed amortization function as in the following calculation.

(continued)		
Press	**Display**	**Comments**
	–1,507	Payment displayed from previous calculation
36 INPUT ■ AMORT	36–36	Specify period 36 through 36 (monthly 36 only)
=	–194.316609	Principal portion of 36th payment
=	–1,312.683391	Interest portion of 36th payment
=	168,870.419441	OSB$_{36}$

The amount of interest paid in the 36th payment is \$1,312.68. The amount of principal paid off in the 36th payment is \$194.32. The outstanding balance immediately following the 36th payment (OSB$_{36}$) is \$168,870.42. As expected, the principal paid and interest paid in the 36th payment total \$1,507, which is the amount of the monthly payment.

PV = \$175,000   End of Term

0   1   2   ...   35   36 months

PMT = \$1,507   \$1,507   \$1,507   \$1,507

Interest = \$1,312.68
Principal = \$194.32

The same pre-programmed amortization function can be used to find the amounts of principal and interest paid over a series of payments, as in questions (d) and (e). In this case, the series of payments will be the entire 36-month loan term, although the amortization function can be used over any series of payments. Questions (d) and (e) can be answered using the following calculation.

(continued)		
Press	**Display**	**Comments**
1 INPUT 36	36	Enter desired series of payments
■ AMORT	1–36	Specify months 1 through 36
=	–6,129.580559	Total principal paid off in payments 1 – 36
=	–48,122.419441	Total interest paid in payments 1 – 36
=	168,870.419441	OSB$_{36}$

The total amount of interest paid over the term of the loan (accumulated interest) was \$48,122.42. The total amount of principal paid off during the loan term was \$6,129.58.

PV = $175,000   End of Term

Interest = $48,122.42

Excel Tip: Interest and Principal Paid

Excel's equivalent functions for calculating interest and principal paid:

- IPMT and PPMT give the interest and principal portion of a given payment, based on the interest rate per period, the period or payment number desired, the number of payment periods, the present value, the future value (OSB at end of term), and the type of annuity (0 = payments at end of period; 1= payments at beginning of period).
       = IPMT(rate,per,nper,pv,fv,type) or
       = PPMT(rate,per,nper,pv,fv,type)

- CUMIPMT and CUMPRINC give the cumulative interest and principal paid over a series of payments, based on the interest rate per period, the number of payment periods, the present value, and the start and ending payment period desired.
       = CUMIPMT(rate,nper,pv,start_period,end_period,type) or
       = CUMPRINC(rate,nper,pv,start_period,end_period,type)

Vendor Financing

Most real estate purchases involve some form of financing, usually secured by a mortgage charge on title. A typical transaction has purchasers arranging their own financing separate from any involvement of the vendor, with a lender of their own choosing. In this transaction, the financing likely has little or any impact on the value of the real property in the transaction, given the purchaser effectively provides the vendor with an "all cash" payment as part of the conveyancing, through a combination of the cash down payment or equity, plus the mortgage funds provided directly from the lender.

Alternatively, there are situations where the purchaser provides only partial cash up front, with the remainder being provided either by assuming the vendor's existing mortgage loan or by the vendor providing partial financing. Vendor financing is often used as a non-price sale incentive. For example, consider a developer who needs to promote the slow sales of building lots. Rather than lowering the price, the developer might instead offer a no-interest or low-interest loan to spur demand. This type of loan would typically be for a short-term and often would be re-sold upon origination to a mortgage investor at a discount, to account for its low rate of interest relative to other market-based loans.

A different advantage of vendor financing might be facilitating sales in situations where purchasers cannot obtain conventional financing. Consider a vendor attempting to sell a residential property in a depressed market, with high unemployment. Purchasers may not be able to qualify for conventional loans, so in order to sell, the vendor may need to consider accepting some cash up front and then the remainder paid back over time. With some funds being paid in future, this vendor is being asked to act as a lender. The loan would usually be secured by a mortgage charge on title. In this scenario, the vendor may consider this a higher-risk loan relative to typical loans, and demand a higher-than-market interest rate.

Where the financing is beneficial to the purchaser, this can create additional value that adds to the package of what the vendor is selling. In other words, rather than selling a parcel of land with a house on it, the vendor is also including beneficial financing. Put another way, the vendor is *selling* the borrower an opportunity to save on future interest payments. And where a purchaser is receiving a benefit, the purchaser should expect to pay for it – it is this potential added "value" that is at the heart of mortgage analysis.

Cash Equivalent Price or Market Value of an Offer

With vendor-supplied (or assumable) mortgage loans, the benefits accruing to the vendor or purchaser can create value above and beyond the real estate. The benefits to the parties may include a higher purchase price when financial benefits are considered or facilitating a purchase that might otherwise not happen if relying only on conventional financing.

Cash equivalent price is a mathematical tool that can be used to at least partially explain these potential benefits. Where the current or "market" rate of interest is greater than the "contract" rate in an assumed or vendor-supplied mortgage, then the offer will actually be worth an amount less than the stated offer price. In other words, the offer is *discounted* to account for the financial benefit the purchaser is receiving – as the purchaser is not just receiving some real estate for the purchase price, but also receiving a beneficial financing package on top of this. The benefit of this financing is subtracted from the apparent purchase price, in order to isolate what the real estate alone is being sold for.

Alternatively, where the current or "market" rate of interest is lower than the "contract" rate in the assumed or vendor mortgage, then the offer will actually be worth an amount more than the stated offer price. In other words, the offer is *bonused* to account for the financial benefit the vendor is receiving – as the vendor is being paid not just for the sale of the real estate, but also for the additional benefit of above-market interest payments (for vendor-financing) or for getting rid of a high interest loan (for an assumed mortgage).

Before accepting any offer requiring the provision of financing at a rate of interest other than the "market" rate, the vendor should be aware of the cash-equivalent price of that offer. The cash equivalent price means that the amount of cash which could reasonably be accepted in lieu of the beneficial financing offer. Figure 1 illustrates the relationship of financing and market value of the offer.

The method used to calculate the cash-equivalent price of an offer is to add the amount of the down payment to the market value of the mortgage(s):

$$
\begin{array}{ll}
& \text{Cash Down Payment} \\
+ & \underline{\text{Market Value of Mortgage}} \\
= & \text{Market Value of Offer (Cash-Equivalent Price of the Offer)}
\end{array}
$$

The obvious question to ask is, "what is the market value of a mortgage?" It is the present value of the future mortgage payments calculated at the market rate of interest. The difference between the market value of the mortgage and its face value can be thought of as the present value of the financial benefits accruing to either the vendor or the purchaser.

Mathematically, the value of assumed or vendor-supplied mortgage is calculated by determining what amount of money an investor would pay today in order to purchase the mortgage contract from the vendor and thereby own the right to receive the future mortgage payments under the loan. The cash the vendor would receive from selling the mortgage contract to such an outside investor is the cash-equivalent value of the mortgage or its market value. In most cases, the interest rate on the loan is below the prevailing market rate, so the investor has to be enticed to purchase the loan by receiving a discount from its face value. As a result, this process is referred to in financial language as "discounting the stream of payments at the market rate of interest".

Figure 1
Financing and the Market Value of the Offer

Offer Involving Below Market Rate Financing

If the market interest rate is higher than the contract rate on the mortgage, then the market value of the offer will be less than the stated offer price.

Offer Involving Market Rate Financing

If the market interest rate is the same as the contract rate on the mortgage, then the market value of the offer will be the same as the stated offer price.

Offer Involving Above Market Rate Financing

If the market interest rate is lower than the contract rate on the mortgage, then the market value of the offer will be greater than the stated offer price.

Summary of the Relationship of Interest Rates and Market Value

Interest rates and market value (present value) are inversely related, when one goes up, the other goes down, and vice versa.

The calculations shown in the remainder of this section outline how to account for financing benefits in cash-equivalent terms or in terms of market value. However, keep in mind that differing interest rates is only one advantage to the parties in these situations. Other advantages could include a sale incentive where typical buyers cannot qualify for conventional financing, a non-price marketing promotion (e.g., no interest for six months), or to help avoid prohibitive prepayment penalties for an existing loan. These non-interest rate benefits may offset the pure financial calculations – the bottom line being the advantage to either vendors or purchasers is purely a matter of negotiation. However, the financial calculations give a solid starting point for analyzing these potential benefits.

Illustration 13

Assume that a prospective vendor listed his property for sale at $245,000, and indicated he might provide some financing to a "qualified purchaser".

Several days later, his real estate broker received a telephone call from a prospective purchaser who wished to view the property immediately. A viewing was arranged and the result was a "full price" offer to purchase the property for $245,000, subject to the vendor taking back a first mortgage for $165,000 at 7% per annum, compounded semi-annually, fully amortized over 25 years with monthly payments. The broker contacted her principal and presented the offer. However, the broker suggested that the offer not be accepted and that it be countered with a similar proposal, except that the mortgage is partially amortized with a 3-year term and a 25-year amortization period.

Upon hearing the details of the offer and the broker's advice to counter offer rather than accept, the vendor was confused. The offer was for the full price, there was a large down payment of $80,000, and the prospective purchaser's income and credit rating were also acceptable. Therefore, the vendor asked the broker to explain her reasons for suggesting the counter offer rather than accepting the purchaser's offer.
Calculate the market value of the mortgage and the market value of the original offer assuming the loan is fully amortized.

Solution

The broker responded with the following detailed analysis:

Analysis of Market Value of the Offer to Purchase (Fully Amortized Loan)

Proposed Offer Price:

	Amount Offered	$245,000
−	Cash Down Payment	− $80,000
	Vendor Mortgage	$165,000

Terms of Proposed Vendor Mortgage:

Face Value:	$165,000
Interest Rate:	$j_2 = 7\%$
Amortization:	25 years
Contractual Term:	25 years
Payments:	Monthly

1. Calculate the Loan Information based on the Contract Rate

(a) Calculate the Equivalent Nominal Rate with Monthly Compounding

Press	Display	Comments
7 ■ NOM%	7	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	7.1225	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	6.900047	Compute nominal rate with monthly compounding

(b) Calculate Monthly Payment
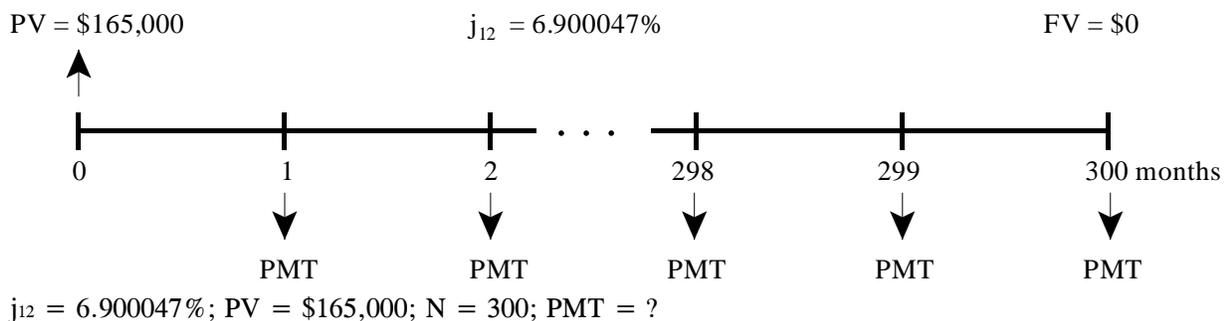
PV = $165,000 $j_{12} = 6.900047\%$ FV = $0

$j_{12} = 6.900047\%$; PV = $165,000; N = 300; PMT = ?

(continued)		
Press	**Display**	**Comments**
	6.900047	j_{12} already stored
165000 PV	165,000	Actual loan amount
25 × 12 = N	300	Enter amortization period in months
0 FV	0	Payments should fully amortize loan over 300 months
PMT	−1,155.685994	Compute monthly payment

The monthly mortgage payment is $1,155.69.

2.   Calculate the Market Value of the Proposed Vendor Mortgage with the Market Rate

In a survey of local lenders, the broker finds that mortgages are available to qualified borrowers at interest rates of 15.5% – 16% per annum, compounded semi-annually. As the prospective borrower desires a fully amortized loan, a rate of $j_2 = 16\%$ is used to determine the market value of the vendor mortgage.

(a) Calculate the Equivalent Nominal Rate with Monthly Compounding

Press	**Display**	**Comments**
16 ■ NOM%	16	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	16.64	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	15.491348	Compute nominal rate with monthly compounding

For the same loan amount and loan terms, this market interest rate would require monthly payments of $2,176.47. This means this purchaser is under-paying in market terms by $1,020.78 per month. Or, put another way, if the purchaser attempted to secure his or her own loan at a 16% interest rate, but only paid $1,155.69 per month, the lender would advance substantially less than $165,000. What we need to calculate is how much less the purchaser could borrow with these payments.

(b) Calculate the Market Value of the Mortgage

$j_{12} = 15.491348\%$; PMT = $1,155.69; N = 300; PV = ?

PV = ? $j_{12} = 15.491348\%$ FV = $0

```
0        1        2        298      299      300 months
```

PMT = $1,155.69 $1,155.69 $1,155.69 $1,155.69 $1,155.69
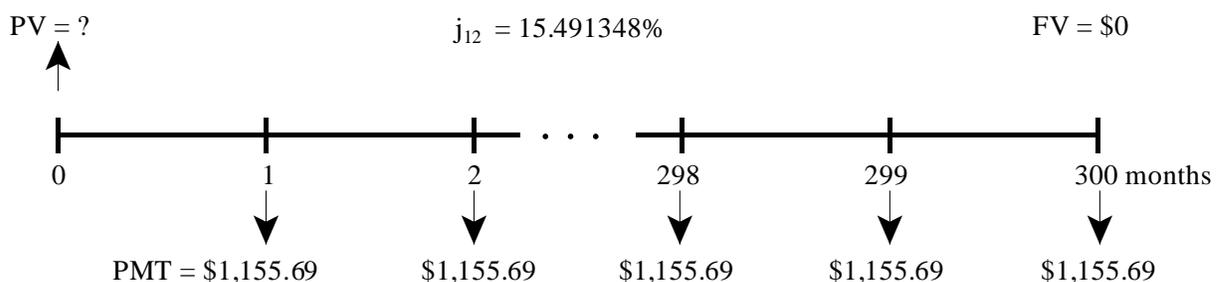
Press	Display	Comments
	15.491348	j_{12} already stored from previous calculation
1155.69 +/– PMT	–1,155.69	Enter rounded payment
PV	87,614.008286	Market value of payment stream

Thus, the payment stream that would result from the proposed vendor mortgage has a present value of $87,614.01 at the rate of interest paid on similar investments. Note, that because all of the other loan information was already entered in the financial keys, once the new interest rate is calculated and the rounded payment is entered, the present value can be calculated without re-entering any other information.

3. Calculate the Market Value of the Offer

If the vendor had accepted the initial offer, he would have received proceeds with a market value (or cash-equivalent price) of $167,614.01 rather than the $245,000 indicated by the stated offer price:

Cash Down Payment	$80,000.00
+ Market Value of the Mortgage	+ $87,614.01
Market Value of the Offer	$167,614.01

If you refer back to Figure 1, you will see that this example illustrates the first relationship. In this example, the mortgage contract rate ($j_2 = 7\%$) was less than the market rate, ($j_2 = 16\%$) which meant the market value of the offer ($167,614.01) was less than the stated offer price ($245,000). In other words, the purchaser has made an offer where they received a substantial financial advantage. The $245,000 offer includes receiving both the underlying real estate and the financing benefit. If the financing benefit is calculated to be worth $77,385.99 ($245,000 face value of the offer less the $167,614.01 market value of the offer), then this implies the purchaser is only offering $167,614.01 for the real estate alone.

Illustration 14

The broker suggested that the vendor counter the offer with a similar arrangement except that the mortgage contains a 3-year term. The market value of the proposed counter offer is determined in a similar manner to the previous example; however, there is also the outstanding balance to consider at the end of the loan term. Calculate the market value of the mortgage and the market value of the offer assuming the loan is partially amortized, i.e., the counter offer.

Solution

Analysis of Market Value of the Proposed Counter Offer (Partially Amortized Loan)

Proposed Counter Offer:

Amount Offered	$245,000
– Cash Down Payment	– $80,000
Vendor Mortgage	$165,000

Terms of Proposed Vendor Mortgage:

Face Value:	$165,000
Interest Rate:	7% per annum, compounded semi-annually
Amortization Period:	25 years
Contractual Term:	3 years
Payments:	Monthly

1.   Calculate the Loan Information with the Contract Rate

(a) Calculate the Equivalent Nominal Contract Rate with Monthly Compounding

Press	Display	Comments
7 ■ NOM%	7	Entered stated nominal rate
2 ■ P/YR	2	Entered stated compounded frequency
■ EFF%	7.1225	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	6.900047	Compute nominal rate with monthly compounding

The equivalent nominal contract rate with monthly compounding is $j_{12} = 6.900047\%$.

(b) Calculate the Monthly Payment and the Outstanding Balance due at the end of the term

   (i)   Payment:
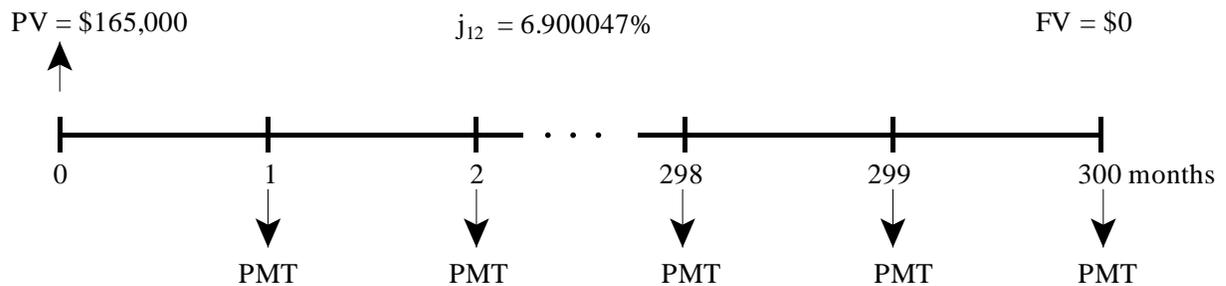
PV = \$165,000 $j_{12} = 6.900047\%$ FV = \$0

(continued)		
Press	**Display**	**Comments**
	6.900047	j_{12} already stored
165000 PV	165,000	Actual loan amount
300 N	300	Enter amortization period in months
0 FV	0	Payments fully amortize loan over 300 months
PMT	−1,155.685994	Compute monthly payment
1155.69 +/− PMT	−1,155.69	Enter rounded payment

(ii) Outstanding Balance:

PV = $165,000 $j_{12} = 6.900047\%$

PMT = $1,155.69 $1,155.69 $1,155.69 $1,155.69 $1,155.69

$FV = OSB_{36} = ?$
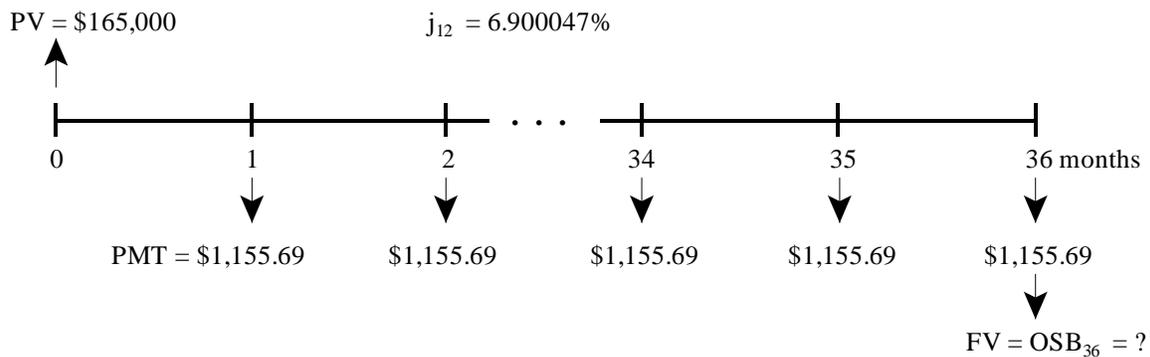
(continued)		
Press	**Display**	**Comments**
36 INPUT ■ AMORT	36–36	
= = =	156,749.516995	Outstanding balance after 36th payment

2.   Calculate the Market Value of Proposed Vendor Mortgage with the Market Rate

Under the terms of the proposed vendor mortgage, the vendor would have the contractual right to receive 36 monthly payments of $1,155.69 as well as, the outstanding balance payment of $156,749.52 at the end of the loan term. In valuing this mortgage proposal, the broker used an interest rate of $j_2 = 15.5\%$ because it is the market rate for 3-year term mortgages.

(a) Calculate the Equivalent Nominal Market Rate with Monthly Compounding

Press	**Display**	**Comments**
15.5 ■ NOM%	15.5	Enter stated nominal rate
2 ■ P/YR	2	Enter stated compounding frequency
■ EFF%	16.100625	Compute equivalent effective annual rate
12 ■ P/YR	12	Enter desired compounding frequency
■ NOM%	15.021956	Compute nominal rate with monthly compounding

The equivalent nominal contract rate with monthly compounding is $j_{12} = 15.021956\%$.

(b) Calculate Market Value of the Mortgage (Present Value of Payments and Outstanding Balance at Market Interest Rate)

N = 36; $j_{12} = 15.021956\%$; PMT = $1,155.69; FV = $156,749.52; PV = ?

PV = ? $j_{12} = 15.021956\%$

PMT = $1,155.69 $1,155.69 $1,155.69 $1,155.69 $1,155.69

$FV = OSB_{36} = \$156,749.52$
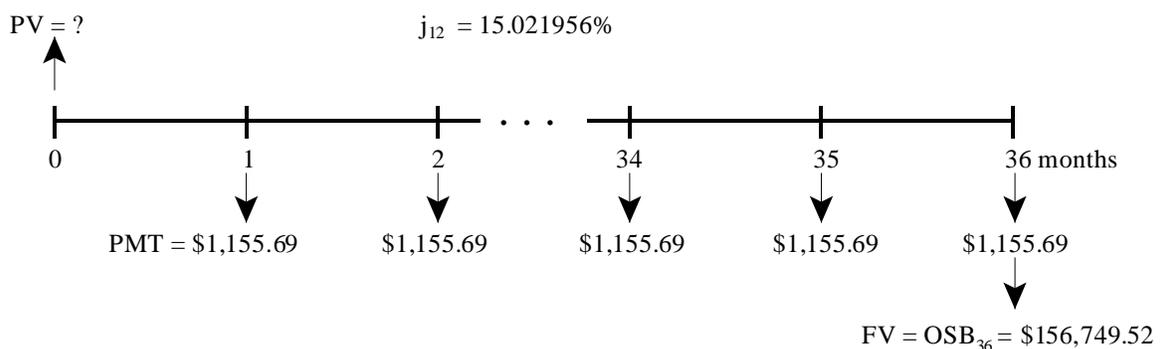
(continued)		
Press	**Display**	**Comments**
	15.021956	j_{12} already stored from previous calculation
36 N	36	36 regular payments to be received
156749.52 +/− FV	−156,749.52	OSB to be received at the end of the term
PV	133,490.058616	Market value of payment stream over the loan term

The market value of the vendor mortgage as proposed in the counter offer is $133,490.06.

3. Calculate the Market Value of the Counter Offer

The counter offer has a market value to the vendor that is much higher than the purchaser's original offer:

	Cash Down Payment	$80,000.00
+	Market Value of Mortgage	+ $133,490.06
	Market Value of Offer	$213,490.06

The counter offer increases the cash value of the transaction from $167,614.01 to $213,490.06, which may better represent the property's true market value and certainly provides more benefit to the vendor, in limiting the length of time the interest advantage is offered to the purchaser.

Illustration 15

Assume that fully amortized mortgages are currently available at 16% per annum, compounded semi-annually, and that 5-year term first mortgages are offered at 15% per annum, compounded semi-annually. Consider the case of a property listed for sale at $75,000. A potential purchaser makes an offer of $73,000, subject to the vendor taking back a $50,000 mortgage at 10% per annum, compounded semi-annually, amortized with monthly payments over 25 years. Calculate the market value of the offer assuming the loan is:
(a) fully amortized
(b) partially amortized over a 5-year

Abbreviated Solution

(a) Calculate the Market Value of the Offer, Subject to Fully Amortized Mortgage

1. Calculate the Loan Information with the Contract Rate

$j_2 = 10\%$
$j_{12} = 9.797815\%$
$PMT = \$447.24$

2. Calculate the Market Value of the Mortgage with the Market Rate

$j_2 = 16\%$
$j_{12} = 15.491348\%$
$PV = \$33,905.71$ (Market Value of Mortgage)

3. Calculate the Market Value of the Offer

	Market Value of Mortgage	$33,905.71
+	Cash Down Payment	+ $23,000.00
	Market Value of Offer	$56,905.71

(b) Calculate the Market Value of the Offer, Subject to Partially Amortized Mortgage

1. Calculate the Loan Information with the Contract Rate

$j_2 = 10\%$
$j_{12} = 9.797815\%$
$PMT = \$447.24$
$OSB_{60} = \$46,996.21$

2. Calculate the Market Value of the Mortgage with the Market Rate

$j_2 = 15\%$
$j_{12} = 14.551655\%$
PV of 60 payments and $OSB_{60} = \$41,789.14$ (Market Value of Mortgage)

3. Calculate the Market Value of the Offer

Market Value of Mortgage		$41,789.14
+ Cash Down Payment	+	$23,000.00
Market Value of Offer		$64,789.14

If the offer was accepted subject to a fully amortized mortgage, the vendor would receive $23,000 cash plus a contract with a face value of $50,000, but with a market value of only $33,905.71. Thus, the offer to the vendor has a cash value of $56,905.71, rather than the stated $73,000. If the vendor accepts the offer subject to the partially amortized loan, the market value of the mortgage would increase to $41,789.14 and the cash value of the offer would increase to $64,789.14, which is still much lower than the $73,000 indicated offer price.

By accepting a mortgage at a rate less than the prevailing market rate, the vendor is, in effect, accepting less for the property than the stated value of the offer.

In summary, a mortgage with a contract interest rate less than the prevailing market rate will have a market value lower than the face value of the loan. Mortgage investors will only buy the mortgage for the amount that the required payments will repay at the current market rate; the contract rate is only of use in determining the payments required under the mortgage.

Further Mortgage Analysis

Vendor financing illustrates how mortgage finance may implicitly have an impact on real property value. This is an important appraisal consideration for any property sale that involves unconventional financing, such as an assumed mortgage or an interest rate buy down. The calculations for these topics are similar to vendor financing, but not exactly the same. Students who want more background on these topics are referred to the BUSI 121 course materials.

Net Present Value (NPV) and Internal Rate of Return (IRR)

Real estate investment analysis requires analyzing the present worth of cash flows received in future. For regular annuities, the PMT key can be used. However, this cannot be used for irregular cash flows. The present value of an investment is the sum of the present values of the net cash flow for each year in the investment, discounted at the investor's yield (discount rate).

Once determined, the present value of the cash flow is compared to the cost of the investment to determine whether the project is a "good investment". As each cash flow receipt is generally different in size (irregular cash flows), each is discounted over the appropriate time period, and the present values of these individual cash flows are then summed to determine the present value of the total investment.

Illustration 16

An investor is considering the purchase of a small apartment building that is forecasted to achieve the following end of year after-tax cash flows:

Year	After-Tax Cash Flow
1	$14,777
2	$21,303
3	$23,285
4	$25,418
5	$19,907 + $345,173

If the investor wants to earn a return of 9% per annum, compounded annually after-tax on equity, what is the maximum amount of equity the investor should be prepared to pay for the real estate investment?

Solution

On a time diagram, the projection would look like the following:
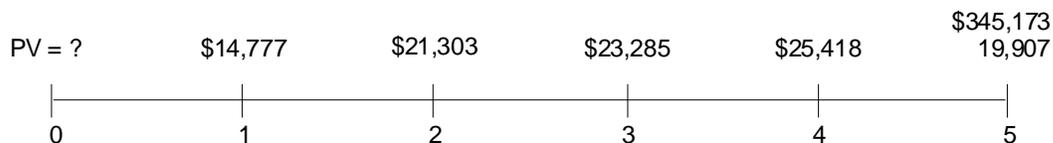
Mathematically, this present value calculation takes the following form:

$$PV = \$14{,}777(1.09)^{-1} + \$21{,}303(1.09)^{-2} + \$23{,}285(1.09)^{-3} + \$25{,}418(1.09)^{-4}$$
$$+ (\$19{,}907 + \$345{,}173)(1.09)^{-5}$$
$$PV = \$304{,}751$$

Press	Display	Comments
■ C ALL	0	Clear all memory registers
9 I/YR	9	After-tax discount rate
1 ■ P/YR	1	
0 CF$_i$	0	No cash flow at time zero
14777 CF$_i$	14,777	Year 1 cash flow
21303 CF$_i$	21,303	Year 2 cash flow
23285 CF$_i$	23,285	Year 3 cash flow
25418 CF$_i$	25,418	Year 4 cash flow
19907 + 345173 = CF$_i$	365,080	Year 5 cash flows
■ NPV	304,751.184488	Present value of investment

It is determined that an investor who requires a minimum after-tax return on equity of 9% per annum, compounded annually should invest no more than $304,751. This maximum is the present value (at $j_1 = 9\%$) of the expected future after-tax cash flows to the equity holder. Note that $304,751 is the present value only if the discount rate is $j_1 = 9\%$. At any other rate, a different present value would be determined. You may wish to confirm that at $j_1 = 7\%$ the present value would be $331,113 and at $j_1 = 12\%$ the present value would be $270,060.

Net Present Value (NPV)

A common variation, or extension, of present value analysis is the *net present value (NPV)*. The net present value is the difference between the present value of all future cash flows and the cost of the investment.

Net Present Value = Sum of the PV of the Net Cash Flows – Investment Cost

Using the investment described in Illustration 16, the net present value is calculated as:

Net Present Value = $304,751 – $235,000

Net Present Value = $69,751

Note that in the previous calculator steps, the first CF$_j$ entered (CF$_0$) was 0. When calculating the present value of future cash flows, the calculator requires that 0 be entered as the investment cost, and that the NPV key be used.

Press	Display	Comments
■ C ALL	0	Clear all memory registers
9 I/YR	9	After-tax discount rate
1 ■ P/YR	1	
235000 +/– CF$_i$	–235,000	Investment cost
14777 CF$_i$	14,777	Year 1cash flow
21303 CF$_i$	21,303	Year 2 cash flow
23285 CF$_i$	23,285	Year 3 cash flow
25418 CF$_i$	25,418	Year 4 cash flow
19907 + 345173 = CF$_i$	365,080	Year 5 cash flows
■ NPV	69,751.184488	NPV of investment

As a general rule, if the net present value, at the investor's required rate of return, is greater than zero, then the investment is acceptable. The net present value is the amount, by which investment value (at a specified required rate of return) exceeds the cost of acquiring the asset.

Illustration 17

A local real estate investment group is considering acquiring a real estate investment. The forecasted annual after-tax net cash flows are as follows:

Year	Net Cash Flow
1	$12,000
2	$12,800
3	$13,600
4	$14,400
5	$15,000
Cost	$45,000

The investment group would like to earn at least 11% per annum, compounded annually after-tax on their investment. Calculate the present value and the net present value of this investment.

Solution

Press	Display	Comments
■ C ALL	0	Clear all memory registers
11 I/YR	11	After-tax discount rate
1 ■ P/YR	1	
0 CF$_i$	0	
12000 CF$_i$	12,000	Year 1 cash flow
12800 CF$_i$	12,800	Year 2 cash flow
13600 CF$_i$	13,600	Year 3 cash flow
14400 CF$_i$	14,400	Year 4 cash flow
15000 CF$_i$	15,000	Year 5 cash flow
■ NPV	49,531.276691	PV of the investment

Present Value at 11%   = $49,531.28

Therefore, the NPV of the investment is the present value of the investment minus the investment cost.

Net Present Value = $49,531.28 – Investment Cost
Net Present Value = $49,531.28 – $45,000
Net Present Value = $4,531.28

Internal Rate of Return (IRR)

The discounted cash flow or present value measure, which provides information regarding the yield on an investment is called the *internal rate of return*. Internal rate of return (IRR) analysis determines the yield on an investment, given its purchase price and expected cash flows. The investor can then compare the investment's calculated IRR against his or her minimum yield expectations.

The internal rate of return provides the yield on an investment, given its purchase price and expected cash flows. The investor can then compare the project's calculated IRR against his or her minimum yield expectations. Mathematically, the IRR is the calculated rate of return that exactly equates the present value of the benefits to the initial acquisition cost. In other words, it calculates the discount rate that would make the net present value equal to zero.

Illustration 18

Using the information from Illustration 16, calculate the internal rate of return on the investment, assuming a cost or initial equity of $235,000.

Solution

Previous analysis determined that the net present value of the investment was $69,751 when the future benefits were discounted at 9%. As one wishes to determine the discount rate at which the net present value equals zero, it is apparent that the interest rate must be higher to reduce the net present value of the cash flows to zero. We deduced that the positive NPV means that benefits exceed costs and therefore the investor's rate of return must exceed 9%. The IRR calculates this exact rate of return, such that the benefits exactly equal costs and the net present value equals zero.

Press	Display	Comments
■ C ALL	0	Clear all memory registers
1 ■ P/YR	1	
235000 +/− CF$_i$	−235,000	Investment cost
14777 CF$_i$	14,777	Year 1 cash flow
21303 CF$_i$	21,303	Year 2 cash flow
23285 CF$_i$	23,285	Year 3 cash flow
25418 CF$_i$	25,418	Year 4 cash flow
19907 + 345173 = CF$_i$	365,080	Year 5 cash flows
■ IRR/YR	15.586354	IRR

The internal rate of return is approximately 15.6%.

An advantage of the internal rate of return measure is that investors are familiar with a percentage rate of return when comparing and analyzing investments. For instance, an investor may accept an investment if the internal rate of return is equal to or exceeds his or her minimum required rate. Here, the internal rate of return of approximately 15.6% exceeds the investor's required rate of 9%.

Excel Tip: NPV and IRR

Excel's equivalent functions for calculating NPV and IRR:

- NPV gives the net present value of an investment based on a discount rate and a series of cash flows.
        = NPV(rate,cash flow 1, cash flow 2,...)

- IRR gives the internal rate of return of an investment based on a series of cash flows and a guess as to the IRR as a starting point.
        = IRR(cash flow 0, cash flow 1,...,guess)

Tips:

- NPV considers the first cash flow as occurring at the end of the first period. As most investments start with the initial investment, as a cash outflow at time zero, this investment cost must be subtracted from the NPV calculation.
        = NPV(rate,cash flow 1, cash flow 2) - cash flow 0

- IRR guess value: this helps IRR operate more quickly, for example if you think the return is 10%, then enter 0.1.

Illustration 19

An investor is considering the purchase of a real estate investment that is expected to produce the following annual after-tax net cash flows. Calculate the present value and net present value at the investor's desired yield of 10% per annum, compounded annually after-tax, and the internal rate of return.

Year	Net Cash Flow
1	$18,500
2	$22,600
3	$27,500
4	$49,850
Cost	$90,000

Solution

Step 1: Calculate the present value and net present value at $j_1 = 10\%$.

Press	Display	Comments
■ C ALL	0	
10 I/YR	10	Discount rate
1 ■ P/YR	1	
90000 +/– CF$_i$	–90,000	Initial investment
18500 CF$_i$	18,500	Year 1 cash flow
22600 CF$_i$	22,600	Year 2 cash flow
27500 CF$_i$	27,500	Year 3 cash flow
49850 CF$_i$	49,850	Year 4 cash flow
■ NPV	205.245543	NPV at j_1 = 10%
+ 90000 =	90,205.245543	PV at j_1 = 10%

Step 2: Calculate IRR:

Press	Display	Comments
■ IRR/YR	10.089622	IRR = 10.09%

Review Problems

1.   Convert each of the following interest rates to the nominal or periodic interest rate requested.

(a) j_2 = 13% i_{sa} = _____
(b) j_{12} = 16% i_{mo} = _____
(c) j_4 = 12.5% i_q = _____
(d) j_{365} = 14% i_d = _____
(e) j_{52} = 6% i_w = _____
(f) i_a = 9.5% j_1 = _____
(g) i_d = 0.03% j_{365} = _____
(h) i_{mo} = 1.692% j_{12} = _____
(i) i_{sa} = 4.8% j_2 = _____
(j) i_w = 0.18% j_{52} = _____

2.   The table below represents investment data collected by an investor for several properties he is considering buying. For each property, the investor has identified three important financial items. Assist the investor by calculating the missing piece of information for each property. Assume in each case that no payments are made (or costs incurred) during the investment horizon and ignore any possible income tax implications.

Property	Purchase Price (PV)	Estimated Future Value (FV)	Holding Period (N)	Interest Rate Earned (j_m)
(a)	$100,000	$185,000.00	6 years	? (j_1)
(b)	$100,000	$185,000.00	6 years	? (j_{12})
(c)	$20,000	?	24 months	j_{12} = 15%
(d)	?	$187,921.00	5 years	j_1 = 24%
(e)	$55,000	$145,931.37	? (years)	j_4 = 20%
(f)	$1,600	$139,294.08	? (years)	j_{12} = 18%
(g)	$27,500	$27,500.00	5 years	? (j_{52})
(h)	$10,000	$20,000.00	1 year	? (j_1)
(i)	$10,000	$20,000.00	2 years	? (j_1)
(j)	$55,179	?	20 years	j_{12} = 12%
(k)	$55,179	?	20 years	j_{12} = 24%

3.   Calculate the missing piece of information for each of the following:

Loan	Purchase Price (PV)	Estimated Future Value (FV)	Payment (PMT)	Holding Period (N)	Interest Rate Paid (j_m)
(a)	$3,576.24	?	$0	6 years	j_1 = 13%
(b)	$4,775.00	$9,061.57	$0	5 years	? (j_1)
(c)	?	$0.00	$200	18 months	j_{12} = 16.5701%
(d)	$1,450.00	$17,980.00	$0	? (semi-annual periods)	j_2 = 11.808758%
(e)	$895.75	$5,000.00	$0	300 months	? (j_{12})
(f)	?	$2,023.73	$0	22 quarters	j_4 = 10.5%
(g)	$650.23	?	$0	3 days	j_{365} = 16.5%
(h)	$ 0.00	?	$1,500	520 weeks	j_{52} = 13.25%

4.   (a) Your client is considering the purchase of a property at the listed price of $179,000. She wishes to earn a minimum of 25.5% per annum, compounded annually. Assume that revenues will equal

costs during the holding period. For what minimum price must this property be sold at the end of the five years in order to realize the 25.5% yield?

(b) If you advise your client the selling price will more likely be $450,000 at the end of five years, what is the maximum price she should pay today in order to still be able to earn the 25.5% yield?

(c) If the client were to sell the property at the end of four years for $450,000, what would be her yield if she paid the listed price?

(d) If this same property could be sold in two parcels, one in two years' time at $150,000 and in five years' time at $200,000 - and the required yield was 27.25%, what is the maximum price that the investor should pay for this property?

5.   Colleen has borrowed $10,000 from Lifetime Trust Company. The loan is interest accruing on which interest is to be charged at a rate of 16% per annum, compounded quarterly (or 4% charged at the end of each 3-month period). How much will Colleen owe at the end of the 5-year term?

6.   Consider an interest only mortgage of $100,000 with an interest rate of 10% per year, compounded monthly (i.e., 0.833333% per month), a term of 10 years, and monthly payments, rounded up to the next higher cent.

(a) How much principal is outstanding after the first year, i.e., after 12 monthly payments?
(b) What is the 13th monthly payment?
(c) What is the 17th monthly payment?
(d) How much principal is owing after the 18th monthly payment?

7.   Mick Bumner wants to buy a boat and spend the rest of his days sailing the South Pacific with his new bride. He figures he can set aside $500 per month towards this goal and that he will need to accumulate $200,000.

(a) If he can earn interest of 16% per annum, compounded monthly, on his monthly deposits, will he have accumulated enough to go at the end of 5 years?

(b) If he decides he wants to go in seven years, how much will he have to deposit per month?

8.   Convert each of the following interest rates to an equivalent monthly rate.

Equivalent j_{12} Rate

(a) $j_2 = 13\%$ _____
(b) $j_2 = 16\%$ _____
(c) $j_2 = 12.5\%$ _____
(d) $j_2 = 14\%$ _____
(e) $j_2 = 20\ 1/8\%$ _____

9.   Convert each of the following interest rates to the equivalent periodic rate requested.

        (a) $j_2 = 11\%$ Equivalent nominal rate with daily compounding   = _____
                                 Equivalent daily periodic rate   = _____

        (b) $j_2 = 19.25\%$ Equivalent nominal rate with annual compounding   = _____
                                 Equivalent annual periodic rate   = _____

        (c) $j_2 = 10\%$ Equivalent nominal rate with weekly compounding   = _____
                                 Equivalent weekly periodic rate   = _____

        (d) $j_4 = 15.6\%$ Equivalent nominal rate with monthly compounding = _____
                                 Equivalent monthly periodic rate = _____

        (e) $j_{12} = 14.64\%$ Equivalent nominal rate with monthly compounding = _____
                                 Equivalent monthly periodic rate = _____

        (f) $j_2 = 12.8\%$ Equivalent nominal rate with daily compounding   = _____
                                 Equivalent daily periodic rate   = _____

        (g) $j_{12} = 13\%$ Equivalent nominal rate with daily compounding   = _____
                                 Equivalent daily periodic rate   = _____

        (h) $j_4 = 15\%$ Equivalent nominal rate with monthly compounding = _____
                                 Equivalent monthly periodic rate = _____

        (i) $j_{365} = 10\%$ Equivalent nominal rate with annual compounding   = _____
                                 Equivalent annual periodic rate   = _____

        (j) $j_2 = 19\%$ Equivalent nominal rate with annual compounding   = _____
                                 Equivalent annual periodic rate   = _____

10.   Calculate the required monthly payment for each of the following loans. (All calculations are to be based on a 25-year amortization period).

	Loan Amount	Interest Rate	Monthly Payment
(a)	$110,000	$j_2 = 19.25\%$	_____
(b)	$6,500	$j_2 = 18\%$	_____
(c)	$37,588	$j_2 = 17.5\%$	_____
(d)	$68,275	$j_2 = 15\%$	_____
(e)	$55,000	$j_2 = 16.5\%$	_____

11.   Complete the following table:

	Mortgage Amount	j_2	Amortization Period (Years)	Monthly Payment
(a)	$105,000.00	19%	25	_____
(b)	$42,000.00	20.5%	10	_____
(c)	$72,500.00	14¼%	30	_____
(d)	$72,712.20	12%	___	$786.00
(e)	$12,250.00	17.2%	___	$284.35
(f)	_____	15.5%	12	$250.00
(g)	_____	13 1/8%	20	$1,800.00
(h)	$125,575.00	___	25	$2,000.00
(i)	$8,750.00	___	8	$175.00

12.   Calculate the outstanding balance at the end of the term for each of the following loans. (All
        calculations are to be based on monthly payments, a 25-year amortization period, and a 5-year term.)

	Loan Amount	Interest Rate	Outstanding Balance (OSB$_{60}$)
(a)	$4,200	$j_2 = 13.25\%$	_____
(b)	$75,000	$j_2 = 10.5\%$	_____
(c)	$83,975	$j_2 = 12\%$	_____
(d)	$92,100	$j_2 = 19\%$	_____
(e)	$18,400	$j_2 = 9\%$	_____

13.   Ted Jones obtained a $12,000 second mortgage at an interest rate of 17% per annum, compounded
        semi-annually. Monthly payments were to be amortized over 20 years, but the outstanding balance was
        to be paid in full at the end of 5 years. Calculate the outstanding balance at the end of the term if:

        (a) payments are rounded up to the next higher dollar
        (b) payments are $175 per month
        (c) payments are $180 per month

14.   A vendor has agreed to provide private financing to a purchaser to help the sale of the property. Under
        this mortgage, the vendor will lend the purchaser $80,000, with a nominal interest rate of 9% per
        annum, compounded semi-annually. The required monthly payments are $711.36. What is the exact
        amortization period of this loan?

15.   Dan Davidson feels he can afford $2,000 per month in mortgage payments. He inquires at the River
        Bank about a mortgage loan in the amount of $117,500. The bank tells him that their current rate is
        11.75% per annum compounded semi-annually on 5-year term mortgage loans, amortized over 25
        years.

        (a) What is the minimum required loan payment on the bank's terms?

        (b) What will the outstanding balance be at the end of the term?

        (c) If the River Bank allows Dan to make payments of $2,000 per month (instead of the payment
             calculated in Part (a)), what will the outstanding balance be at the end of the term?

        (d) If Dan made a prepayment of $10,000 at the end of the second year and payments of $2,000 per
             month for the full five years, what would his outstanding balance be at the end of the term? You
             may presume that his contract has no penalty for prepayment. (Hint: Find OSB$_{24}$, deduct $10,000,
             enter result as new loan amount (PV), then find OSB$_{36.}$)

16.   Sally can afford a maximum of $850 per month in mortgage payments. She has approached two lending
        institutions.

        (a) Joe's Bank is willing to lend mortgage funds to be fully amortized by monthly payments over a
             20-year amortization period at an interest rate of 14.25% per annum, compounded semi-annually.
             Given the maximum payment Sally can afford, what is the maximum loan she could receive from
             Joe's Bank?

        (b) Sam's Credit Union will lend funds at an interest rate of 14% per annum, compounded monthly,
             to be fully amortized by monthly payments over a 30-year amortization period. What is the
             maximum loan amount which Sam's Credit Union could offer Sally?

17.   A borrower has arranged a mortgage loan in the amount of $75,000 with a 30-year amortization period and a 3-year term. The interest rate on the loan is 12.5% per annum compounded semi-annually. Monthly payments are rounded up to the next higher dollar. Calculate:

         (a) the required monthly payment
         (b) the outstanding balance at the end of the term

18.   A borrower has arranged for a $64,500 mortgage loan. The interest rate will be 12.375% per annum, compounded quarterly, the payments will be made monthly, and the amortization period will be 20 years.

         Calculate the outstanding balance if the loan has:

         (a) a 2-year term
         (b) a 4-year term
         (c) a 5-year term

19.   Erin Baxter took out a mortgage loan which had a 5-year term. She borrowed $63,500 at an interest rate of 9.25% per annum, compounded semi-annually; made monthly payments based on a 25-year amortization. Calculate the total interest paid and the total principal paid off over the 5-year term.

20.   A prospective purchaser offers to purchase a property for $195,000 providing the vendor takes back a mortgage in the amount of $150,000. This vendor financing is to be written at $j_2 = 12\%$ per annum, and is to be fully amortized over a 25-year period with constant monthly payments. If the market rate of interest on a fully amortized mortgage is $j_2 = 15\%$, what is the market value of this offer?

         (a) Convert the contract rate of interest to an equivalent nominal rate with monthly compounding.
         (b) Calculate the monthly payment required under the $150,000 mortgage.
         (c) Calculate the equivalent nominal rate with monthly compounding for the market rate of $j_2 = 15\%$.
         (d) Calculate the present value of the vendor mortgage at the market rate of interest.
         (e) Calculate the market value of the offer.

21.   A prospective purchaser has made an offer to purchase a property for $115,000, given that the vendor will provide financing at the rate of 1.2% per month. The vendor take-back is to be in the amount of $75,000, providing for constant monthly payments. The mortgage is to be written over a 25-year period, but will have a term of only 5 years. What is the market value of this offer if the market rate of interest on a partially amortized mortgage is 18% per annum, compounded monthly?

         (a) Calculate the monthly payment required under the $75,000 mortgage.
         (b) Calculate the outstanding balance at the end of the term.
         (c) Using the market rate of interest, calculate the total present value of the vendor financing arrangements.
         (d) Calculate the market value of the offer.

22.   A proposed vendor take-back mortgage has a face value of $114,000 at an interest rate of 12.34% per annum, compounded semi-annually. Monthly payments are rounded to the next higher dollar, the amortization period is 15 years, and the term is 2 years. Current market mortgage rates are 14.75% per annum, compounded semi-annually. Calculate the market value of the vendor mortgage.

23.   A vendor has agreed to grant a purchaser a vendor mortgage in order to facilitate the sale of his house. The mortgage is to be in the amount of $75,000 and is to be fully amortized over a period of 20 years at an interest rate of 8.5% per annum, compounded semi-annually. Payments are to be made monthly. If the purchaser has offered to purchase the house for a total of $115,000, calculate the market value of the offer. Assume current market interest rates are 10.75% per annum, compounded semi-annually.

24.   A prospective purchaser has requested vendor financing in the amount of $50,000. This financing
       arrangement is to be fully amortized over a 20-year period by monthly payments. The interest rate to
       be paid is 0.85% per month. What is the market value of this vendor take-back mortgage if the market
       interest rate on a fully amortized mortgage is 1.15% per month?

       (a) Calculate the monthly payment.
       (b) Calculate the market value of the mortgage.

       Assume instead that the vendor financing arrangement above had a 5-year term. What would be the
       market value of the vendor take-back?

       (c) Calculate the outstanding balance at the end of the term.
       (d) Calculate the market value of the vendor take-back mortgage.

25.   A vendor has accepted a $137,000 first mortgage "take-back" to facilitate the sale of a commercial
       property. The loan calls for annual payments to amortize the loan over 20 years at an interest rate of 8%
       per annum, compounded annually.

       (a) Calculate the required payment.

       (b) Calculate the market value of the mortgage if interest rates at the time the vendor sold the property
           were 12% per annum, compounded semi-annually,

26.   A vendor agrees to take-back a mortgage of $93,500 at a rate of 7% per annum, compounded
       semi-annually, amortized over 15 years, but with a 2-year term. Payments are to be rounded to the next
       higher $100 and made monthly.

       (a) Calculate the required payment.
       (b) Calculate the outstanding balance at the end of the term.
       (c) Calculate the market value of the mortgage if similar mortgages are currently available at 9% per
           annum, compounded semi-annually.

27.   Based on the projected annual cash flows shown below, calculate the net present value (NPV) of this
       investment if the investor requires a yield of $j_1 = 10\%$.

Year	Cash Flow
1	$10,800
2	$10,800
3	$12,275
4	$14,350
5	$10,800
6	$9,750
7	$13,540
Cost	$47,500

28.   Janie Brown of Janie's Waffle Hut, has realized a large profit in the past three years, and now wishes to invest this money in an income producing property. Janie's financial advisor, Ryan, has detailed several possibilities that may meet her requirements. The forecasted cash flows and acquisition costs for each property are listed below.

Year	M	N	O
1	$150,000	$0	$250,000
2	150,000	0	-50,000
3	150,000	650,000	500,000
Costs (Today)	300,000	320,000	300,000

(a) What is the internal rate of return on Investment M?
(b) What is the internal rate of return on Investment N?
(c) What is the internal rate of return on Investment O?

29.   Stephanie and Kit have heard that Japa-Dogs are a big hit in Vancouver, so they want to set up a cart in Edmonton. They have figured out the cost and estimate the following annual net cash flows, after paying expenses. They plan to sell the cart at the end of 5 years, for a profit. Calculate the present value and net present value at $j_1 = 9\%$, and the internal rate of return.

Year	Net Cash Flow
1	$21,480
2	$14,270
3	$13,280
4	$14,271
5	$62,000
Cost	$55,325

Solutions

1.   (a) i_{sa} =   6.5%
        (b) i_{mo} =   1.333333%
        (c) i_q =   3.125%
        (d) i_d =   0.038356%
        (e) i_w =   0.115385%
        (f) j_1 =   9.5%
        (g) j_{365} =   10.95%
        (h) j_{12} =   20.304%
        (i) j_2 =   9.6%
        (j) j_{52} =   9.36%

2.   (a) j_1 =   10.797158%
        (b) j_{12} =   10.297021%
        (c) FV   = $26,947.02
        (d) PV   = $64,101.31
        (e) N   = 20 quarters = 5 years
        (f) N   = 300 months = 25 years
        (g) j_{52} =   0%
        (h) j_1 =   100%
        (i) j_1 =   41.421356%
        (j) FV   = $601,040.22
        (k) FV   = $6,394,624.52

3.   (a) FV   = $7,445.56
        (b) j_1 =   13.670041%
        (c) PV   = $3,168.26
        (d) N   =   43.887986
             semi-annual periods
        (e) j_{12} =   6.897877%
        (f) PV   = $1,144.41
        (g) FV   = $651.11
        (h) FV   = $1,622,312

4.   (a) $557,277.69
        (b) $144,541.94
        (c) 25.918595%
        (d) $152,578.48

5. $21,911.23

6.   (a) $100,000
        (b) $833.34
        (c) $833.34
        (d) $100,000

7.   (a) No, he will only have $45,517.76
        (b) $1,305.75 per month

8.   (a) 12.661289%
        (b) 15.491348%
        (c) 12.186387%
        (d) 13.608312%
        (e) 19.32967%

9.   (a) j_{365} =   10.709724%
            i_d =   0.029342%
        (b) j_1 =   20.176406%
            i_a =   20.176406%
        (c) j_{52} =   9.767194%
            i_w =   0.187831%
        (d) j_{12} =   15.401483%
            i_{mo} =   1.283457%
        (e) j_{12} =   14.64%
            i_{mo} =   1.22%
        (f) j_{365} =   12.409187%
            i_d =   0.033998%
        (g) j_{365} =   12.932378%
            i_d =   0.035431%
        (h) j_{12} =   14.816311%
            i_{mo} =   1.234693%
        (i) j_1 =   10.515578%
            i_a =   10.515578%
        (j) j_1 =   19.9025%
            i_a =   19.9025%

10.   (a) $1,715.05
        (b) $95.32
        (c) $537.29
        (d) $850.81
        (e) $745.66

11.   (a) $1,617.58
        (b) $802.67
        (c) $850.12
        (d) 20.000004
        (e) 5.499793
        (f) $16,641.26
        (g) $155,718.89
        (h) 19.700895%
        (i) 19.179328%

12.   (a) $ 4,040.57
        (b) $70,794.51
        (c) $80,162.48
        (d) $90,627.51
        (e) $17,133.18

13.   (a) $11,380.47
        (b) $11,012.02
        (c) $10,551.45

14.   N = 239.992246

15. (a) $1,191.98
     (b) $111,971.77
     (c) $46,905.73
     (d) $32,820.60

16. (a) $68,979.93
     (b) $71,737.72

17. (a) $783
     (b) $74,077.64

18. (a) $62,795.19
     (b) $60,619.80
     (c) $59,316.36

19. Principal paid off = $4,226.61
     Interest paid = $27,947.19

20. (a) 11.710553%
     (b) $1,547.85
     (c) 14.551655%
     (d) $124,211
     (e) $169,211

21. (a) $925.85
     (b) $72,747.50
     (c) $66,235.48
     (d) $106,235.48

22. $109,604.62

23. $104,423.38

24. (a) $489.16
     (b) $39,800.76
     (c) $45,005.29
     (d) $43,779.25

25. (a) $13,953.76
     (b) $101,918.65

26. (a) $900
     (b) $84,203
     (c) $90,342.15

27. NPV = $9,425.17

28. (a) 23.38%
    (b) 26.64%
    (c) 48.08%

29. PV = $92,377
    NPV = $37,052
    IRR = 27.89%