Executive Summary. While a useful tool, the internal rate of return (IRR) has limitations that require understanding. This Point of View reflects on the usual list of these with brief explanations but then delves more deeply into one overlooked in the literature. Jensen’s (1966) Inequality has been known for some time but few realize how it introduces a mathematical inconsistency into the simulation of the IRR. The computation of the bias that results from passing a linear operator through a curved function is explored.

With the publication of Crean (2005) one is reminded of the industry’s long love–hate relationship with the internal rate of return (IRR). This Point of View is intended to briefly enumerate several IRR problems that have enjoyed considerable commentary in the literature and discuss one that has long been troublesome but less discussed. Because of ubiquitous fast, cheap desktop computing power, recently this latter conundrum has become more important.

When IRR Must Be Used with Caution

In addition to the reinvestment problem Crean (2005) examines, there are a handful of other difficulties with the IRR.

The “No Solution Problem”

At times one may obtain no solution at all. Finding the IRR is the process of locating the root of a polynomial. The root is sometimes referred to as the “zero” because the solution is found by solving for the point on the X-axis where function crosses (at a zero value). The only closed form solution for the polynomial we know as the IRR is via the quadratic equation. Such a closed form solution places a restriction on the number of cash flows to no more than two. When the number of cash flows exceeds two, the common case in a real estate context, finding a solution involves an iterative search along the X-axis for zero point. If this search begins at an inopportune place where the outer search limits do not conscribe the zero, the search
comes up empty. The cure for this is purely mechanical. The search device, a computer, spreadsheet or hand-held calculator, is given a “guess,” which is closer to the zero sought. As the device searches in a range that does contain the zero in question, it ultimately identifies it.

The “Multiple Solution Problem”
This is a variation on the no solution problem above. As we will discover below, the IRR is a convex function. As such, it is possible under the right conditions to cross the X-axis twice. The result is that there are two roots to the polynomial and both are mathematically correct answers. Usually the device used to “discover” the IRR stops when it finds the first root. Again, depending on the starting point, any root may be found. The “right conditions” are those in which a sign change occurs in the middle of the cash flow series. This always produces multiple answers, not a comforting result for the real estate analyst. The cure is to perform a discounting process on the offending (usually negative) cash flow(s) in the middle of the series, which reduce the cash flow(s) to zero and move its (their) economic impact(s) to the beginning of the series. When the present discounted value of such cash flow(s) are combined with the initial investment, the remaining cash flows in the middle are all of the same sign, the function crosses the X-axis only once, there is one root, and the IRR to be found is unique. One is faced with the choice of the discount rate for this process, an issue related to Crean’s (2005) observations about reinvestment rates.

The “Ranking Problem”
A more serious issue is the false signal given by the IRR when contemplating mutually exclusive investments. In corporate finance, the term used is “Capital Rationing,” a fancy moniker that merely means you do not have enough money to pursue all opportunities you identify, so you must choose. Real estate, while also constrained by budgetary issues, introduces its own unique variation to this problem. Consider a piece of vacant land. Local zoning codes may allow development into several uses. Once the land is committed to a certain use, there is no turning back. Building out the project involves choosing between mutually exclusive alternatives. We wish to rank them in an order of preference. Because the IRR is sensitive to the timing of cash flows, it is possible that net present value (NPV) analysis will rank one project higher than the other but IRR will rank them in the opposite way. There is no good answer for this problem other than understanding how it occurs. There is a point of indifference between the two decision tools, a crossover point, where the IRR and NPV produce the same solution. The choice of the hurdle rate for the NPV is what governs whether the inconsistency exists. When the NPV hurdle rate is to the left of the crossover point on the X-axis, the IRR and NPV will always conflict. When the hurdle rate occurs after the crossover point, the two decision rules will be consonant.

The “Scale Problem”
The easier of the two regards return and has to do with which alternative results in the most terminal wealth (given that wealth maximization is the primary goal). If you are asked whether you prefer an IRR of 10% or 50% and told the risk does not vary, you might naively choose the 50% return. But when you are given the additional information that you are restricted to investing a small dollar amount at 50% and no such restriction exists at 10%, your decision might well change because the nominal dollar amount of terminal wealth can vary substantially based on the permitted size of the initial investment.

The second part of the scale problem has to do with the difference between risk pooling and risk sharing. The simple example here is how you would view the prospects of betting $1,000 with an equal chance of receiving back either zero or $3,000. From an IRR perspective, the former outcome offers a −100% return and the latter a +200% return. Intuition might suggest that you could lower your risk by diversifying, making a thousand bets, under the same payoff conditions, of $1,000 each. This is a fallacy known as risk pooling and, in fact actually increases risk because the size of what is
at risk increases. What lowers risk—true diversification—is risk sharing. Under these conditions, you would make a 1,000 $1 bets, each with an equal chance of nothing or $3. The size of what is at risk remains the same from the original case but the distribution of outcomes is broader. A little time with the equations of portfolio management will convince you that standard deviation of terminal wealth falls with this latter strategy. Both of these issues are peculiar to the use of percentage (IRR) as a measure and disappear when nominal dollar amounts are substituted.

The “Simulation Problem”
This is the problem the present effort wishes to highlight.

Jensen’s Inequality
The IRR is a curved function of cash flow. Well known real estate data problems often forces practitioners to simulate outcomes to introduce risk analysis to the discussion. As the IRR is often the return metric of choice, it is tempting to simulate it. Because of its curvature, expectations formed from simulating the IRR necessarily produce a troublesome bias arising from what is known as Jensen’s Inequality.

This year is the 100th anniversary of Jensen’s (1906) paper on curved functions that left us with some important insight about these mathematical constructs. For the moment, we shall focus on the convex function. Jensen’s Inequality holds that, for convex functions, the function of the expectation is always less than or equal to the expectation of the function. This is formally defined as:\(^1\)

A function, \(f\), is convex in the interval \(I\) if and only if the following inequality is satisfied for all \(x_1, x_2, \ldots, x_n\) in \(I\) and for all \(\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0\) with \(\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0:\)

\[
f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n).
\]

The message is: since the expectation is a linear operator, it is bad form to employ it with curved functions. Graphically, Exhibit 1 shows how the IRR function at all times lies on or below a line from which we may compute its expectation.\(^2\)

Exhibit 2 provides a numerical illustration of this phenomenon, computing the bias by subtracting the IRR of the expected cash flows (the function of the expectation) from the average IRR produced from two different cash flow series (the expectation of the function).

By design, the cash flows illustrated in Exhibit 2 have a particularly wide range. This is important in that the amount of the bias is governed by the degree of variance. An illustration with a smaller range of cash flow difference would produce a bias, which might be considered trivial. This highlights the misery practitioners face. Simulation is used to model variation. The more variation, presumably the more simulation is helpful. But as variation grows, so does the bias. Hence, simulation only produces a consistent answer when there is no variation, a condition under which simulation is unnecessary.\(^3\)

The story becomes murkier when the terminal cash flow is less than the initial investment. Under those conditions, the bias is negative, meaning that the IRR function is then concave. Because anticipating less than a full return of initial investment is rare, this situation is encountered less frequently in a real estate setting.

What is an analyst to do? The solution for the Jensen bias problem is the same as the solution for all the other IRR problems: Do not use IRR, use NPV. Interestingly, NPV is a linear function of cash
### Exhibit 2
Numerical Illustration of IRR Function
Jensen's Inequality-Two Outcome Case

<table>
<thead>
<tr>
<th>Potential Cash Flow #1</th>
<th>Potential Cash Flow #2</th>
<th>Potential Cash Flow #3*</th>
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<td>( n )</td>
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<td>Reversion</td>
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- IRR: 12.1118%
- Average IRR: 27.3116%
- IRR of Averaged CFs: 26.8307%
- Bias: 0.4809%
- IRR of Averaged CFs: 26.83%

*Expectation of Annual Cash Flows #1 and #2
flow. As such, there is no bias when taking its expectation. If one is intent on using IRR, reporting the bias with the simulation results is a minimum. When the bias is trivial, a client may elect to ignore it. Most important, the analyst should be aware of this phenomenon and recognize that a mathematical inconsistency is being introduced into the process when the IRR is simulated.

Conclusion

While the discussion has highlighted a number of problems with IRR, more exist. Others have found fault with the IRR on theoretical grounds. Young (1979) showed that the variant financial management internal rate of return provided a specious decision tool. Robichek and Myers (1966) complained that a single discount rate should not capture both risk and the time value of money.

We are probably stuck with the IRR as a performance tool because it provides a useful yardstick by which to compare returns on real estate investments with other alternatives. Clients demand it and practitioners have become familiar with it. Knowing its limitations constitutes important due diligence. Disclosing those limitations increases the credibility of the analyst.

Endnotes

1. Concave functions are defined similarly by reversing the inequality sign.

2. Those who remember the early days may recall constructing the IRR from the Ellwood Tables. The procedure involved finding two points that bracketed the true IRR on either side and then making a linear interpolation by hand to find a point in the middle. Because the interpolation was linear, we were always cautioned to disclose to our client that the IRR was approximate. Exhibit 1 displays a similar situation calling for similar treatment.

3. One may view this illustration in an interactive setting with animated output at http://mathestate.com/hands_on/simulateIRR.jsp. A link is provided there to permit the reader to download worksheet used to produce Exhibit 2.

4. With an exponent in the denominator, at first glance this does not appear to be so. However, that exponent is an index. So in the usual case of equally-spaced cash flow intervals, the denominator is just a number.

5. Explaining what it means will represent a challenge. Most clients eyes will glaze over at a discuss of the difference between IRR and NPV, so the analyst must be prepared to be patient and well prepared when the inevitable question “What does that mean?” arises.

6. This indictment is more about how we view risk than about the IRR itself and affects the NPV approach equally.

References


